

# Warped and eccentric discs around black holes

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**Abstract.** Accretion discs around black holes in X-ray binary stars are warped if the spin axis of the black hole is not perpendicular to the binary orbital plane. They can also become eccentric through an instability involving a resonance with the binary orbit. Depending on the thickness of the disc and the efficiency of dissipative processes, these global deformations may be able to propagate into the innermost part of the disc in the form of stationary bending or density waves. We describe the solutions in the linear regime and discuss the conditions under which a warp or eccentricity is likely to produce significant activity in the inner region, which may include the excitation of quasi-periodic oscillations.

**Keywords:** accretion discs – black holes – hydrodynamics

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## INTRODUCTION

Accretion discs involve approximately Keplerian motion around a massive central object, and the general solution allows for nested orbits with smoothly varying inclination and eccentricity. The shape of a warped or eccentric disc evolves slowly under the action of stresses in the disc and external forces that deviate from that of a Newtonian point mass. Coherent precession of discs is usually possible in binary stars.

Warped and eccentric discs may be relevant to a wide variety of phenomena in X-ray binaries. Long-period modulations such as superhumps and superorbital variability may be attributable to the precession of global warping or eccentric modes that are not forced but are excited by various instabilities. Sufficiently large discs encounter a resonance at the location where the angular velocity of the disc is three times that of the binary orbit and may become eccentric as a result [1, 2]. This effect is well known in cataclysmic variable stars of mass ratio  $q \lesssim 0.3$  [e.g. 3] where, as well as giving rise to superhumps, the eccentricity enhances the viscous dissipation and significantly affects the outburst dynamics. Similar processes should occur in most X-ray binaries with black hole primaries, and indeed superhumps are reported in an increasing number of such systems [4–8]. A warping instability involving radiation forces [9] is more likely to occur with neutron star primaries [10] but may also be seen in GRS 1915+105, which has a very large disc [11]. Stationary warped discs also arise whenever there is a misalignment between the orbital angular momentum of the binary and the spin angular momentum of the central object [12], a knowledge of which is communicated to the disc through general relativistic (gravitomagnetic) or magnetic torques [13, 14].

The innermost parts of discs around black holes and neutron stars depart significantly from Keplerian motion. Indeed, the rapid relativistic precession of elliptical or inclined orbits has often been discussed in connection with quasi-periodic oscillations (QPOs)

in X-ray binaries, and may best be studied within the context of eccentric or warped discs. There is an apparent conflict between the description of the outer part of the disc, which supports slowly precessing global deformations, and that of the non-Keplerian inner region. In this paper we attempt to make this connection and to describe how a global deformation of the disc in the form of a stationary or slowly precessing warp or eccentricity may be able to propagate inwards under some conditions, activating the inner region and possibly exciting trapped oscillations that may explain high-frequency QPOs in accreting black holes. The excitation mechanism itself has been studied by Kato [15, 16] and by Ferreira and Ogilvie [17, see also this volume].

## LOCAL ANALYSIS

A small warp or eccentricity can be considered as a perturbation of a standard (circular and coplanar) disc. Such a disc supports a variety of wave modes, having a dependence on time and azimuth of the form  $\exp(im\phi - i\omega t)$ . In the simplest case of a strictly isothermal disc, a local dispersion relation

$$k^2 H^2 = \frac{(\hat{\omega}^2 - \kappa^2)(\hat{\omega}^2 - n\Omega_z^2)}{\hat{\omega}^2 \Omega_z^2} \quad (1)$$

can be derived [18], which relates the Doppler-shifted wave frequency  $\hat{\omega} = \omega - m\Omega$  to the radial wavenumber  $k$ . The integers  $m$  and  $n \geq 0$  are the azimuthal and vertical mode numbers, while  $H$  is the vertical scaleheight of the disc and  $\Omega$ ,  $\kappa$  and  $\Omega_z$  are the orbital, epicyclic and vertical oscillation frequencies characteristic of circular orbits in the given potential or metric. All of these quantities depend on the radius  $R$  at which the dispersion relation is evaluated. The local dispersion relation of a more general disc model can be calculated numerically. For this description to be accurate, the wavelength  $\lambda = 2\pi/k$  should be much less than  $R$ .

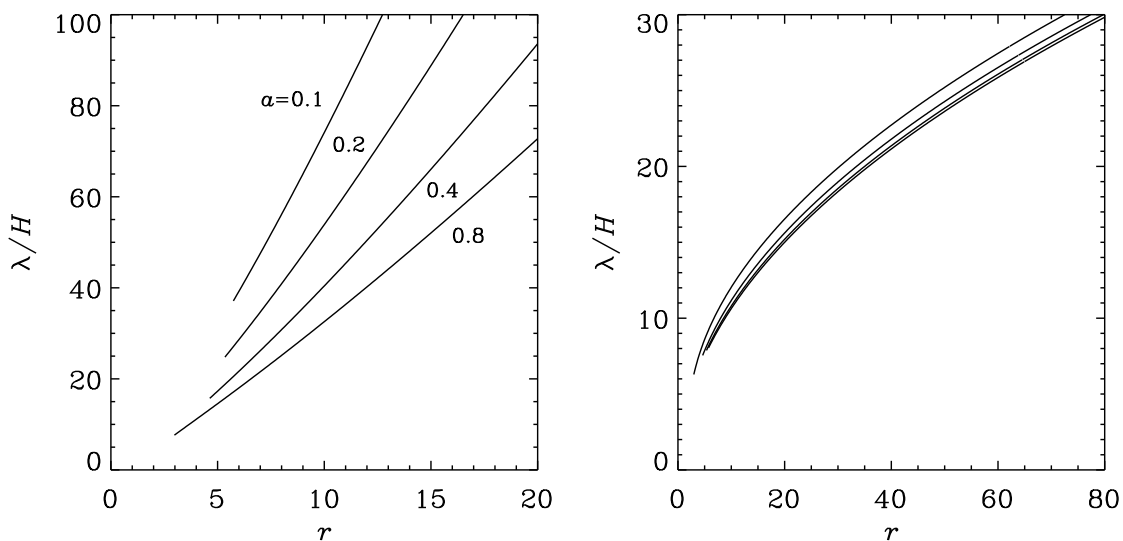
Within this context, a warp corresponds to  $(m, n) = (1, 1)$  (vertical motion independent of  $z$ ) and an eccentricity to  $(m, n) = (1, 0)$  (horizontal motion independent of  $z$ ). The dispersion relation shows that the warp takes the form of a propagating bending wave ( $k^2 > 0$ ) when  $(\omega - \Omega)^2 > \max(\kappa^2, \Omega_z^2)$  or  $< \min(\kappa^2, \Omega_z^2)$ , while the eccentricity takes the form of a propagating density wave when  $(\omega - \Omega)^2 > \kappa^2$ .

## SECULAR THEORIES

A complementary description is provided by theories that consider a warp or eccentricity that varies on a length-scale much longer than  $H$  and on a time-scale much longer than  $\Omega^{-1}$ . Small-amplitude warps are governed by the equations

$$\Sigma R^2 \Omega \left[ \frac{\partial W}{\partial t} - i \left( \frac{\Omega^2 - \Omega_z^2}{2\Omega} \right) W \right] = \frac{1}{R} \frac{\partial G}{\partial R}, \quad (2)$$

$$\frac{\partial G}{\partial t} - i \left( \frac{\Omega^2 - \kappa^2}{2\Omega} \right) G + \alpha_W \Omega G = \frac{PR^3 \Omega}{4} \frac{\partial W}{\partial R} \quad (3)$$



**FIGURE 1.** Local wavelength of a stationary warp (*left*) or eccentricity (*right*), in units of the local vertical scaleheight  $H$ , plotted versus the radius in gravitational units, for black holes with spin parameters  $a = 0.1, 0.2, 0.4$  and  $0.8$ . The disc is terminated at the marginally stable orbit in each case.

[e.g. 19], where  $W$  describes the amplitude and phase of the inclination of the disc at radius  $R$  and time  $t$ ,  $G$  refers to a horizontal torque communicated by Reynolds stresses,  $\Sigma$  is the surface density and  $P = \Sigma H^2 \Omega_z^2$  is the vertically integrated pressure. These equations describe propagating bending waves with essentially the same local dispersion relation as in the previous section. (The theories overlap when  $H \ll \lambda \ll R$ , which is possible in a thin disc.) They also allow for viscous (i.e. turbulent) damping of the warp, parametrized using a dimensionless number  $\alpha_W$ , which is equivalent to the usual Shakura–Sunyaev parameter [20] if the disc has an isotropic effective viscosity.

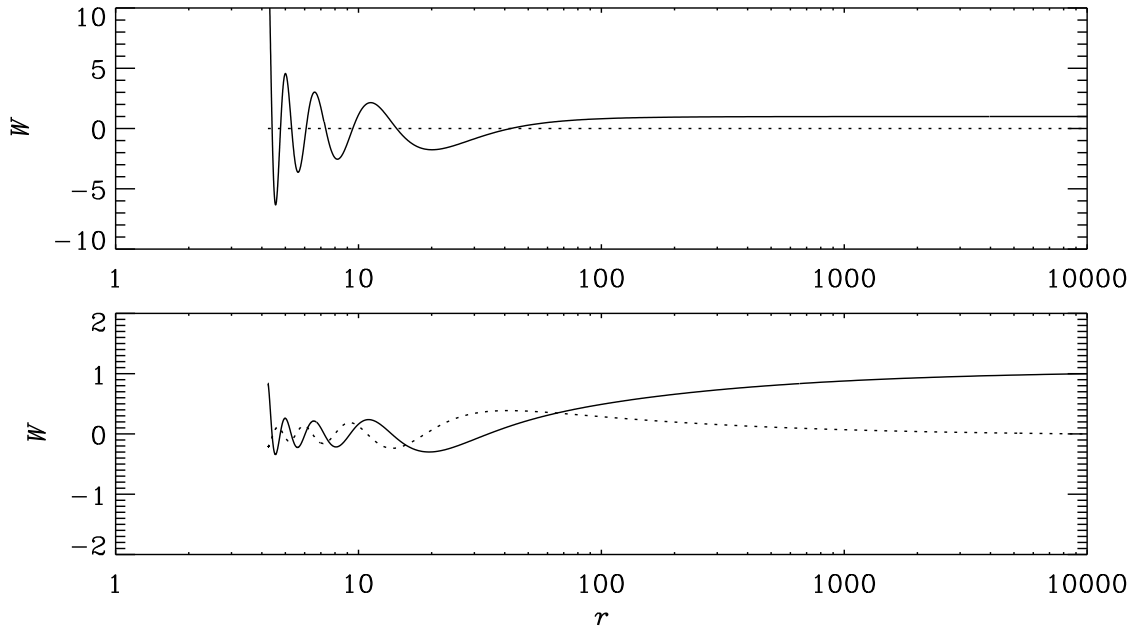
The simplest equation describing a small eccentricity is

$$-2i\Sigma R^2 \Omega \frac{\partial E}{\partial t} = \frac{1}{R} \frac{\partial}{\partial R} \left[ (\gamma - i\alpha_E) P R^3 \frac{\partial E}{\partial R} \right] + R \frac{\partial P}{\partial R} E + \Sigma R^2 (\Omega^2 - \kappa^2) E \quad (4)$$

[e.g. 21], where  $E$  describes the amplitude and phase of the eccentricity of the disc at radius  $R$  and time  $t$ ,  $\gamma$  is the adiabatic exponent and  $\alpha_E$  parametrizes the viscous damping. For short wavelengths, this equation also agrees with the local dispersion relation of the isothermal disc when  $\gamma = 1$  and  $\alpha_E = 0$ . This equation is based on a two-dimensional approximation and neglects many of the complications of a shear viscosity such as viscous overstability [22]. More sophisticated theories, including nonlinearity and all viscous or viscoelastic effects, are available in the literature [23, 24].

## STATIONARY DEFORMATIONS

A global warp or eccentricity precesses at only a fraction of the binary orbital frequency, so the wave frequency  $\omega$  is completely negligible compared to  $\Omega$ ,  $\kappa$  and  $\Omega_z$  in the inner

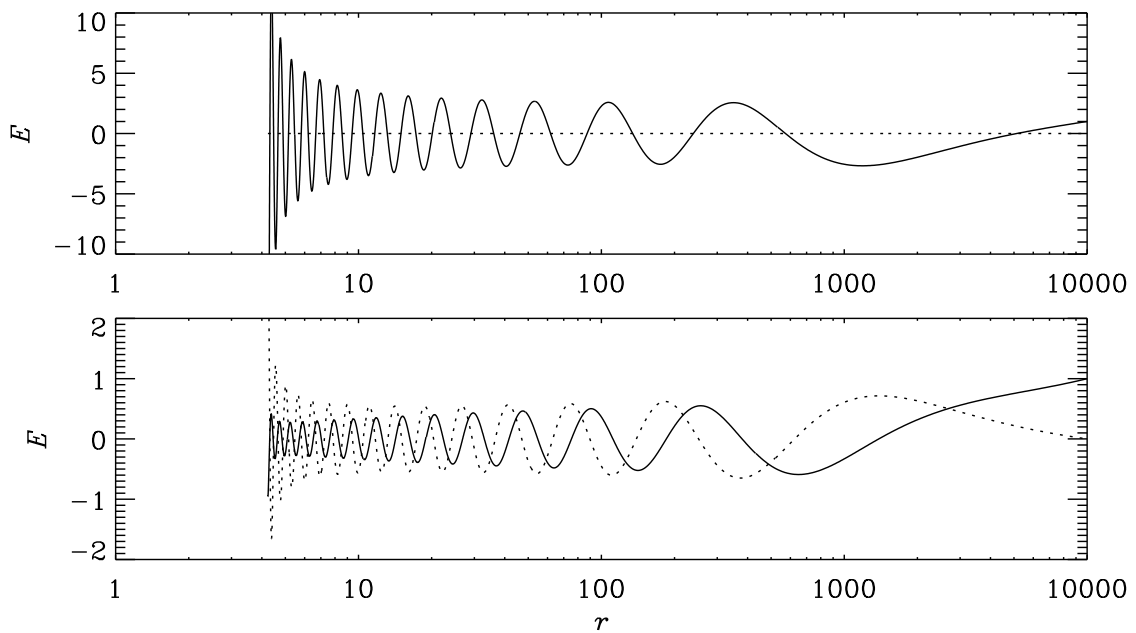


**FIGURE 2.** Stationary warp in a disc around a black hole with  $a = 0.5$ ,  $\varepsilon = 0.02$  and with  $\alpha_W = 0$  (*top*) and  $0.05$  (*bottom*). Real and imaginary parts of  $W$  are plotted as solid and dotted lines. The amplitude is scaled such that  $W \rightarrow 1$  at large  $r$ .

part of the disc. Setting  $\omega = 0$ , we find the local radial wavenumber to be given by  $k^2 H^2 = (\Omega^2 - \kappa^2)(\Omega^2 - \Omega_z^2)/\Omega^2 \Omega_z^2$  in the case of a warp and  $k^2 H^2 = (\Omega^2 - \kappa^2)/\gamma \Omega_z^2$  for an eccentricity. Using the expressions for  $\Omega$ ,  $\kappa$  and  $\Omega_z$  for the Kerr metric [25] and taking  $\gamma = 5/3$ , we find  $\lambda/H$  as a function of the dimensionless radius  $r = Rc^2/GM$  and the spin parameter  $a$  (see Fig. 1). For  $a > 0$ , both  $W$  and  $E$  propagate at all radii, with wavelengths everywhere significantly longer than  $H$ . Of all the wave modes described by the local dispersion relation, these are the most credible in a turbulent disc because of their relatively long wavelengths.

It is also possible to predict how the amplitudes of the deformations scale with radius. In the absence of viscous damping, a WKB analysis of the secular theories shows that  $|W| \propto R^{1/8}(\Sigma H)^{-1/2}$  and  $|E| \propto R^{1/4}(\Sigma H)^{-1/2}$ . We apply these results to a steady accretion disc in the regime dominated by gas pressure and Thomson opacity, in which (assuming  $\alpha = \text{constant}$ )  $\Sigma \propto f^{3/5} R^{-3/5}$  and  $H \propto f^{1/5} R^{21/20}$ , where  $f = 1 - (R_{in}/R)^{1/2}$  [20]. Then  $|W| \propto f^{-2/5} R^{-1/10}$  and  $|E| \propto f^{-2/5} R^{1/40}$ , implying a very mild dependence of the amplitude on radius. However, the gradients  $dW/dR$  and  $dE/dR$  do increase sharply at small  $R$ , because of the rapidly decreasing wavelength.

When dissipation is taken into account, these solutions are modified by viscous attenuation. Our interest is in whether a deformation of the outer part of the disc can propagate into the inner region with non-negligible amplitude. Noting that  $H/R$  is almost independent of  $R$  in the above model, we define the constant parameter  $\varepsilon$  by  $H/R = \varepsilon f^{1/5} r^{1/20}$ . Then the logarithm of the attenuation factor for a complete crossing of the disc scales approximately with  $\alpha_W/\varepsilon$  or  $\alpha_E/\varepsilon$ .



**FIGURE 3.** Stationary eccentricity in a disc around a black hole with  $a = 0.5$ ,  $\varepsilon = 0.02$  and with  $\alpha_E = 0$  (*top*) and  $0.05$  (*bottom*). Real and imaginary parts of  $E$  are plotted as solid and dotted lines.

The numerical solutions in Fig. 2 confirm this behaviour. In the absence of dissipation, the bending wave reflects perfectly from the stress-free inner boundary and sets up a standing wave. Significant attenuation is found when  $\alpha_W$  exceeds  $\varepsilon$ , in which case only a wave with inward group velocity is seen. If  $\alpha_W$  is increased much further, the oscillations are no longer apparent. Only in this case does the inner part of the disc lie in the equatorial plane of the black hole as suggested by Bardeen and Petterson [13]. Note that  $W$  tends to a constant at large  $r$ , which corresponds to the inclination of the outer part of the disc with respect to the black hole's equator. The oscillatory structure of the warp has been noted before [19, 26]. Traces of it may already have been detected in numerical simulations [27].

The behaviour of the eccentricity is similar (Fig. 3) except for the shorter wavelength (still everywhere much longer than  $H$ ). Again, unless  $\alpha_E$  is several times greater than  $\varepsilon$ , the eccentricity can reach the inner region.

Several caveats accompany these solutions. Radiation pressure, which is more important at higher accretion rates, thickens the inner part of the disc, increases the wavelength, and reduces the attenuation. Viscous overstability may cause the eccentricity to grow, rather than decay, as it propagates inwards. Nonlinearity may be very important. The relevant damping coefficients  $\alpha_W$  and  $\alpha_E$  are not generally equal to  $\alpha$  and remain relatively poorly understood. In addition, the time for the warp or eccentricity to propagate into the inner region and to establish the steady solutions shown here can be long.

## CONCLUSIONS

Accretion discs around black holes in X-ray binary stars may commonly be warped or eccentric. Depending on the thickness of the disc and the efficiency of dissipative processes, these global deformations may be able to propagate into the innermost part of the disc in the form of stationary bending or density waves. This is most likely to occur when the disc is hotter and thicker, when the wavelengths are longer and the viscous attenuation is less severe. Under these conditions the inner region may be activated and trapped oscillations may be excited through nonlinear mode couplings [15–17].

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