1 What is high energy astrophysics?

High energy astrophysics has many different interpretations. In the most narrow sense this is the type of observations involving high energy photons, primarily X-rays and gamma-rays. From a more physical point of view one usually means the study of objects which involve either extreme conditions, like high energies, temperature or densities, or photons and particles with high energies. Therefore, one includes such objects as cosmic rays, which traditionally was the first area of high energy astrophysics, high energy neutrinos, X-rays, gamma-rays, from the 'detector' point of view. From a more astrophysical point of view this includes supernovae, supernova remnants, neutron stars, black holes, binary X-ray sources, gamma-ray bursts, active galactic nuclei, radio jets, clusters of galaxies. In addition to these fairly exotic objects also more ordinary objects like ordinary stars and galaxies are also emitters of non-thermal radio emission and X-rays. Also the neutrinos from the sun are usually included. Some areas like high energy neutrino astronomy and gravitational waves are not mature enough to be useful as diagnostics of these objects, but will probably be extremely useful in a few years. This also applies to different types of dark matter detection, where e.g., the GLAST gamma-ray satellite will be extremely interesting for detection of gamma-rays resulting from e.g. dark matter annihilation in the Galaxy or from discrete sources like the sun or the galactic center.

Classes of objects: Stellar remnants, active galactic nuclei, clusters of galaxies,

Supplement to Longair Chap. 13 and 14

2 Equations of stellar structure

See e.g., Kippenhan and Weigert

Mass conservation

\[
\frac{dM}{dr} = 4\pi r^2 \rho(r) \quad (2.1)
\]

Hydrostatic equilibrium

\[
dF = 4\pi r^2 (p(r + dr) - p(r)) = -4\pi r^2 \rho(r) \frac{GM(r)}{r^2} \quad (2.2)
\]

or

\[
\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad (2.3)
\]
Energy conservation

\[ \frac{dL}{dr} = -4\pi r^2 \epsilon(r) \]  

(2.4)

where \( L \) is the luminosity (total energy loss per unit time) and \( \epsilon \) is the energy generation rate per volume.

Finally we need an equation describing the transport of energy from the center to the surface. Suppose first that the energy transport is by diffusion of the photons. In the interior the gas is so optically thick that the mean free path, \( \lambda \), is small compared to the dimensions of the system (e.g., the star). The photon will therefore perform a random walk, or diffusion. The radiation field will therefore be almost, but not completely, isotropic. We will therefore derive a diffusion equation describing the flow of radiation as a result of the small temperature gradient.

Consider an almost isotropic radiation field, as is the case in the interior of a star, and let us calculate the energy flowing through a surface area \( dA \) from a direction \( \theta \) to the radial direction. The projected area is then \( \cos \theta dA \). These will on the average come from a distance \( \lambda \) equal to the mean free path. The fraction of photons coming from this direction is \( \frac{2\pi \sin \theta d\theta}{4\pi} \).

The total energy through \( dA \) will then be

\[ dE = \frac{1}{2} \sin \theta d\theta u(r + dr) \cos \theta dAdt = \frac{1}{2} \sin \theta d\theta u(r - \lambda \cos \theta) \cos \theta dAdt \]  

(2.5)

Averaging over all directions

\[ dE = c/2 \int_0^\pi \sin \theta \cos \theta u(r - \lambda \cos \theta) d\theta dAdt \]  

(2.6)

or making a Taylor expansion

\[ dE = c/2 \int_0^\pi \sin \theta \cos \theta u(r) d\theta dAdt - c/2\lambda \int_0^\pi \sin \theta \cos^2 \theta \frac{du}{dr} d\theta dAdt \]  

(2.7)

or since \( u(r) \) is nearly isotropic (diffusion approximation!), the first term is zero and

\[ dE = -\frac{c\lambda}{3} \frac{du}{dr} dAdt \]  

(2.8)

In terms of the flux, \( F = \frac{dE}{dAdt} \), we have

\[ F = -\frac{c\lambda}{3} \frac{du}{dr} \]  

(2.9)

The mean free path is \( \lambda = 1/\rho \kappa \) where \( \kappa \) is the opacity and \( \rho \) the density. The radiation density in an optically thick atmosphere is \( u = aT^4 \), so

\[ F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr} \]  

(2.10)
The luminosity is given by \( L = 4\pi r^2 F \), so

\[
\frac{dT}{dr} = -\frac{3\kappa \rho L(r)}{16\pi acr^2 T^3}
\]  

(2.11)

Equations (2.1),(2.3),(2.4), and (2.11) define the structure of a star dominated by radiation transport. Under certain conditions (especially high temperature sensitivity of the energy generation or low temperature (high opacity)), resulting in very steep temperature gradients if the transport occurs only by radiation, the energy transport occurs instead by convection. A nearly adiabatic temperature gradient is then set up,

\[
\frac{dT}{dr} = -\frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{p} \frac{dp}{dr}
\]  

(2.12)

where \( \Gamma_2 \) is an effective adiabatic index.

Most of the physics of the star and its evolution is determined by the opacity, \( \kappa \), the energy generation, \( \epsilon \), and the equation of state, \( p(\rho, T) \).

## 3 The equation of state

The pressure, \( P \), of a non-degenerate, perfect gas with temperature \( T \) is given by

\[
P = k n T
\]  

(3.1)

where \( n \) is the number of particles per volume and \( k \) is Boltzmann’s constant, \( k = 1.38 \times 10^{-16} \) ergs K\(^{-1}\). In terms of the density this can be written as

\[
P = \frac{k}{m_u \mu} \rho T
\]  

(3.2)

where \( \mu \) is the mean mass per particle and \( m_u \) the atomic mass unit, \( 1.667 \times 10^{-24} \) g.

For a gas of fermions the number density of particles is

\[
n = \frac{8\pi}{h^3} \int_0^\infty f(p)p^2 dp
\]  

(3.3)

where \( p \) is the momentum, \( f(p) = 1/[\exp(E - \mu)/kT + 1] \) is the Fermi-Dirac distribution, and \( \mu \) is the chemical potential. The factor \( 4\pi p^2 dp/h^3 \) is the phase space factor, and another factor of two comes from the spin of the electrons. For a fully degenerate gas \( f(p) = 1 \) for \( p < p_F \) and \( f(p) = 0 \) for \( p > p_F \), allowing us to solve for \( p_F \)

\[
p_F = \left( \frac{3h^3 n_e}{8\pi} \right)^{1/3}
\]  

(3.4)
This can be written in terms of the Fermi energy using $E_F = \sqrt{p_F^2 c^2 + m_e^2 c^4}$.

For a non-relativistic gas $E_F = p_F^2 / 2m_e$, while for a relativistic gas $E_F = p_F c$.

The pressure, $P$, is given by

$$P = \frac{1}{3} \int_0^\infty v(p) p \frac{dn(p)}{dp} dp$$

(3.5)

where $dn/dp = 8\pi f(p)p^2/h^3$. For a fully degenerate gas we get

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} v(p)p^3 dp .$$

(3.6)

Now $p = m_e v / \sqrt{1 - v^2/c^2}$, or

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4}{\sqrt{m^2 + p^2/c^2}} dp .$$

(3.7)

For simplicity we consider the non-relativistic and ultra-relativistic limits separately. The transition occurs when $p_F \approx m_e c$. Using $\rho = \mu_e m_p n_e$ this occurs at

$$\rho_r = 9.7 \times 10^5 \mu_e \text{ g cm}^{-3}$$

(3.8)

For a non-relativistic gas $p \ll mc$ and Eq. (3.7) shows that

$$P = \frac{8\pi}{15mh^3} p_F^5 .$$

(3.9)

With $p_F$ from Eq. (3.4) we finally get

$$P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2 m}{n} 5/3 ,$$

(3.10)

which is the equation of state for a non-relativistic, completely degenerate gas. In terms of the density we get in cgs units

$$P = 1.00 \times 10^{13} \mu_e^{-5/3} \rho^{5/3} .$$

(3.11)

In the opposite limit of an ultra-relativistic gas we obtain in the same way from Eq. (3.7)

$$P = \frac{2\pi c}{3h^3} p_F^4 .$$

(3.12)

and

$$P = \frac{1}{8} \left( \frac{3}{\pi} \right)^{1/3} hc n^{4/3} ,$$

(3.13)
which is the equation of state for an ultra-relativistic, completely degenerate gas. Note that the adiabatic index in this case is $4/3$, while in the non-relativistic case it is $5/3$. In cgs units

$$P = 1.24 \times 10^{15} \mu_{e}^{-4/3} \rho^{4/3}. \quad (3.14)$$

For a non-relativistic gas the boundary between degeneracy and perfect gas equations of state is obtained by setting the non-degenerate pressure (Eq. (3.2)) equal to the degenerate, given by Eq. (3.10)

$$\frac{T}{\rho^{2/3}} = 1.2 \times 10^{6} \frac{\mu}{\mu_{e}^{5/3}}. \quad (3.15)$$

Similarly, if the gas is relativistic one finds that the boundary between degeneracy and perfect gas equations of state is given by

$$\frac{T}{\rho^{1/3}} = 1.5 \times 10^{7} \frac{\mu}{\mu_{e}^{4/3}}. \quad (3.16)$$

The density when degeneracy sets in depends on the mass of the particle and temperature as $n_{\text{deg}} \propto m^{3/2}T^{3/2}$ in the non-relativistic case and $n_{\text{deg}} \propto m^{3}$ in the relativistic. Therefore, even if the electrons are degenerate, the ions are usually non-degenerate. The total pressure is then given by

$$P = P_{e} + P_{\text{ion}} \quad (3.17)$$

where $P_{e}$ is given by either Eq. (3.10) or Eq. (3.13) and $P_{\text{ion}}$ by Eq. (3.1). In the strongly degenerate case the ion pressure is much smaller than that of the degenerate electrons, and can usually be neglected.

Finally, the boundary between an ideal gas pressure and that of radiation dominated pressure is given by For a non-relativistic gas the boundary between degeneracy and perfect gas equations of state is give by

$$\frac{RT_{p}}{\mu} = \frac{aT^{4}}{3} \quad (3.18)$$

or

$$\frac{T}{\rho^{1/3}} = 3.2 \times 10^{7} \mu^{-1/3} \quad (3.19)$$

In Fig. 1 we show the different regions defined by Eqns. (3.8), and (3.15) – (3.19) in the $\rho - T$ plane.
In Longair a derivation of the Chandrasekhar mass is given in a rigorous way. Because of its importance it may be of interest also to derive it in a 'quick and dirty' way, without losing too much of the physics.

Let us consider the structure of a completely degenerate core. The equation of hydrostatic equilibrium gives

\[
\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2} \tag{4.1}
\]

or approximately

\[
P \approx \frac{GM\rho}{r} \tag{4.2}
\]

Now \( \rho \approx 3M/(4\pi R^3) \), so

\[
P \approx GM^{2/3} \rho^{4/3} \tag{4.3}
\]

Here we can use the expression for the equation of state derived above. Alternatively, we note that the pressure is given approximately by

\[
P \approx \frac{1}{3} n_e p v \tag{4.4}
\]

where \( p \) is the 'typical' momentum of the electrons, and \( n_e \) the electron density and \( v \) the velocity. An estimate of \( p \) can be obtained from the uncertainty relation \( \Delta p \Delta x \approx \hbar \), or since \( n_e \approx 1/\Delta x^3 \),

\[
P \approx \frac{1}{3} \hbar n_e^{4/3} v \tag{4.5}
\]
If the electrons are relativistic then \( v = c \), and we have

\[
P \approx \frac{1}{3} \hbar n_e^{4/3} c \tag{4.6}
\]

which is apart from a numerical factor Eq. (3.13).

If the electrons are non-relativistic \( v \approx \hbar n_e^{1/3} / m_e \), again using the uncertainty relation. Therefore,

\[
P \approx \frac{1}{3} \frac{\hbar^2}{m_e} n_e^{5/3} c \tag{4.7}
\]

To convert from electron density to total mass density we use \( n_e = Z \rho / A m_p \approx 1/2 \rho / m_p \) for a gas dominated by heavy elements. This together with Eq. (4.7) in Eq. (4.3) gives in the non-relativistic case

\[
R \approx \frac{\hbar^2}{G m_e M^{1/3}} \left( \frac{Z}{A m_p} \right)^{5/3} \tag{4.8}
\]

This shows that as the mass of the star increases the radius decreases. Therefore, as the mass increases the density will increase as \( \rho \propto R^{-6} \), and the electrons will therefore become relativistic at \( \sim 10^6 \text{ g cm}^{-3} \).

The mass – radius above can be tested directly if masses and radii can be determined from e.g. binary motions and spectroscopy, including the gravitational redshift. In Figure 2 we show a comparison between observations of a sample of white dwarfs and the theoretical mass - radius relation for different interior composition.

When the electrons are relativistic the equation of state is given by Eq. (4.6) which together with Eq. (4.3) gives an expression independent of radius,

\[
M_{\text{Ch}} \approx \left( \frac{c \hbar}{G} \right)^{3/2} \left( \frac{Z}{A m_p} \right)^2 \tag{4.9}
\]

This is the Chandrasekhar mass apart from numerical factors. A more accurate value is

\[
M_{\text{Ch}} = 2.018 \frac{(3\pi)^{3/2}}{2} \left( \frac{c \hbar}{G} \right)^{3/2} \left( \frac{Z}{A m_p} \right)^2 = 5.836 \left( \frac{Z}{A} \right)^2 M_\odot \tag{4.10}
\]

For \( Z/A = 0.5 \) we get \( M_{\text{Ch}} = 1.46 M_\odot \).
Figure 2: Comparison between observations and the theoretical mass-radius relation for different interior composition. (Provencal et al. (1998))
5 Solar neutrinos

5.1 Predicted fluxes

In Fig. 3 the expected fluxes from the 'standard' solar model are shown. Note the different threshold energies of the different experiments.

\[
\nu_e + ^{37}\text{Cl} \rightarrow ^{37}\text{Ar} + e
\]  

(5.1)

Figure 3: Predicted neutrino spectrum from the sun (Bahcall 2004)

5.2 Neutrino experiments

Homestake Gold Mine

1500 m underground. 615 tons of cleaning fluid, \(C_2Cl_4\).

\[
\nu_e + ^{37}\text{Cl} \rightarrow ^{37}\text{Ar} + e
\]  

(5.1)
threshold 0.8 MeV. Detect radioactive decay of $^{37}$Ar. $\sim 15$ reactions per month!

\[ \nu_e + ^{37}\text{Cl} \rightarrow e + ^{37}\text{Ar} \]  
(Homestake, SD)

Figure 4: The Homestake neutrino detector.

The average measured rate of the Homestake experiment is $2.56\pm 0.25$ SNU, while the predicted is $8.1 \pm 1.2$.

**Superkamiokande**

1000 m underground. 50,000 ton water Cerenkov detector

\[ \nu_e + e^- \rightarrow \nu_e + e^- \]  
(5.2)

threshold 5 MeV. Cherenkov light from scattered electrons 15 events/day! Directional information!

**SAGE**

50 tons of gallium!

\[ \nu_e + ^{71}\text{Ga} \rightarrow ^{71}\text{Ge} + e \]  
(5.3)

Threshold: 0.233 MeV. **Sensitive to p + p neutrinos!**

Most recent result: 70.8 (+5.3, -5.2) (+3.7, -3.2) SNU
GALLEX

30 tons of gallium in the Gran Sasso laboratory in Italy.

Measured rate $77.5 \pm 8$ SNU. SSM prediction: $129 +8/-6$ SNU

The SNO experiment

Since the book of Longair was published there has been considerable progress in terms of new experiments on solar neutrinos. The most spectacular of these is the SNO experiment (Sudbury Neutrino Observatory). While all the other experiments are only sensitive to the electron neutrino this is sensitive to all three types of neutrinos.

This experiment is based on a detector containing 1000 tons of heavy water, $^2\text{H}_2\text{O}$, nearly 2000 m underground close to Sudbury, Ontario. The incoming neutrinos can react in two different ways with the deuterium nuclei.

In the first reaction the $\nu_e$ can disintegrate the deuteron, resulting in a change of the charge of the nucleons

$$\nu_e + ^2\text{H} \rightarrow p + p + e$$

The electron can be detected from its Cherenkov radiation. This reaction only occurs for the electron neutrino.
The most interesting reaction is

$$\nu_x + ^2\text{H} \rightarrow p + n + \nu_x$$

(5.5)

which occurs by neutral currents. Here $x$ stands for either $e, \mu$ or $\tau$. This reaction, which only disintegrates the deuteron without any changes of the charges, occurs with all three neutrino types. Furthermore, the probability is the same for all three types. The neutrons liberated are captured by $^{35}\text{Cl}$ nuclei, which converts it to $^{36}\text{Cl}$, with emission of gamma-rays.

The amazing result from this detector is that while the charged current reaction gave a similar result as the other neutrino experiments, $\sim 30\%$ of the expected rate, the neutral current reaction gave nearly exactly the rate predicted by the standard solar model! The situation is summarized in Fig. 9.

This result clearly demonstrated that the electron neutrinos changed flavor into the other two types on the way from the sun to us. Of the original electron neutrinos only about a third oscillated back to electron neutrinos, while the rest were converted to $\mu$ and $\tau$ neutrinos. Because the
other experiments are not sensitive to these this explains the more than four decade long puzzle.

**KamLAND**

Besides the SNO experiment this is the most interesting. Here a detector in Kamioka, containing 1000 tons of liquid scintillator, detects neutrinos from nuclear reactors at different distances, 80–350 km, in Korea and Japan. The neutrino flux from these reactors passes through a long column density of mass. During this passage the neutrinos will mix by the MSW effect and only a fraction of the original emitted electron anti-neutrinos will be detected as such, the rest being in the tau- and mu- flavor states. The neutrinos collide in the detector with protons, converting them to neutrons, which are inducing radioactive isotopes, which decay.

\[ \bar{\nu}_e + p \rightarrow n + e^+ \]  \hspace{1cm} (5.6)

The experiment therefore checks the survival probability as function of distance from the reactor.

As shown in Fig. 8 the flux is indeed suppressed relative experiments done at short distances. The full line shows the predicted rate for mixing parameters determined from solar observations.

The fact that there is excellent agreement between the prediction and the experiment shows that neutrino oscillations are confirmed as an explanation of the 'neutrino problem'.
Figure 8: Neutrino flux as function of distance from the reactor. The line gives the expected rate, assuming that the neutrino oscillations have the mixing angles and mass differences derived from the solar observations.

Summary

The neutrino fluxes corrected for neutrino mixing for the different reactions are summarized below relative to the predicted rates from the standard solar model:

\[
\begin{align*}
\text{Flux}(\text{pp}) & = 1.02 \pm 0.02 \pm 0.01 \times \text{theory} \\
\text{Flux}(8\text{B}) & = 0.88 \pm 0.04 \pm 0.23 \times \text{theory} \\
\text{Flux}(7\text{Be}) & = 0.91^{+0.24}_{-0.62} \pm 0.11 \times \text{theory}
\end{align*}
\]

Explains the more than four decade long puzzle, and shows that we indeed understand why the sun shines!
Figure 9: Neutrino fluxes as measured by different experiments. Note the agreement with the neutral current SNO measurements and the predicted value (Bahcall 2004).
6 Pulsars

6.1 What are the pulsars?

Some basic facts: $\sim 1000$ pulsars detected. Of these $\sim 30$ millisecond pulsars, most of these in binary systems. The periods are in the range 0.00156 - 8.5 secs. The pulse periods are stable to the same level as atomic clocks, approximately one part in $10^{12}$ or better. A fact that is important for the formation process of the pulsars is that they in general have very high space velocities, $100 - 1000$ km s$^{-1}$. It is likely that this was either cause by an asymmetric explosion in connection to the supernova collapse and explosion, or because the supernova occurred in a binary system, which was disrupted by the explosion.

Today we know that pulsars are rapidly rotating neutron stars. It is, however, instructive to review some of the arguments in favor of this. Let us first calculate the maximum velocity before break-up by requiring that the centrifugal force should be less than the gravitational force on the surface,

$$\frac{V^2}{r} < \frac{GM}{R^2} \tag{6.1}$$

But $P = 2\pi R/V$, and $M \approx 4\pi R^3 \rho/3$ so

$$\rho > \frac{5.6 \times 10^8}{P^2} \text{ g cm}^{-3} \tag{6.2}$$

The maximum density of a white dwarf is $\sim 10^{10}$ g cm$^{-3}$. The fact that pulsars with as short periods as a few milli-seconds have been observed clearly rules out white dwarfs. Instead, these periods indicate densities close to nuclear, $\gtrsim 10^{14}$ g cm$^{-3}$. This is what is expected for a neutron star. Also pulsations of a white dwarf is excluded, based on both the short period and the stability of the pulses.

6.2 Pulsar slowing down (Longair p. 100-104)

Far from the magnetic pulsar the magnetic field can be approximated by a dipole field with magnetic moment $p_m$. If this dipole field has an angle $\alpha$ with the rotation axis (defined to be the z-axis), the magnetic moment will vary as

$$p = p_0 (\cos \alpha \mathbf{e}_z + \sin \alpha \cos \Omega \mathbf{e}_x + \sin \alpha \sin \Omega \mathbf{e}_y) \tag{6.3}$$
Figure 10: An updated version of the $\dot{P}$ versus $P$ plot. (Lorimer & Kramer)
The total energy loss is
\[ \frac{dE}{dt} = -\frac{2\ddot{p}^2}{3c^3} \] (6.4)
which is the same as the energy loss from an electric dipole (see e.g., Jackson or Landau-Lifshitz Eq. 71.5). Using Eq. (6.3) for \( \ddot{p} \) we get
\[ \frac{dE}{dt} = -\frac{2p_0^2\Omega^4 \sin^2 \alpha}{3c^3} \] (6.5)
Note that there is no energy loss if the dipole is aligned to the rotational axis.

The energy loss by radiation is taken from the loss of rotational, kinetic energy, given by \( E_k = \frac{I\Omega^2}{2} \), where \( I \) is the moment of inertia. For a uniform sphere this is given by
\[ I = \frac{2}{5}MR^2 \] (6.6)
Therefore setting \( dE_k/dt = dE/dt \) we get
\[ \frac{d}{dt} \frac{I\Omega^2}{2} = I\Omega \frac{d\Omega}{dt} = -\frac{2p_0^2\sin^2 \alpha}{3c^3} \Omega^4 \] (6.7)
The dipole field is given by
\[ B_r = \frac{2p_0 \cos \theta}{r^3} \]
\[ B_\theta = \frac{p_0 \sin \theta}{r^3} \] (6.8)
(see e.g., Jackson 1962, Eq. 5.41). Here \( \theta \) is the polar angle, now from the magnetic axis, and \( r \) the radial direction. For a current \( I \) in a circle with radius \( a \) we have \( p_0 = \pi a^2 I/c \).

Let us know assume that we can estimate the mass and radius of the neutron star, as \( M \approx 1 \text{ M}_\odot \) and \( R \approx 12 \text{ km} \). We can then if we can measure \( \Omega \) and \( d\Omega/dt \), from Eq. (6.7) estimate \( p_0 \). Because \( B \approx p_0/R^3 \) at the surface, we can then estimate the magnetic field at the surface.

As an example we take the Crab where \( P = 0.033 \text{ s} \) (i.e., \( \Omega = 190 \text{ Hz} \)) and \( \dot{P} = 4.2 \times 10^{-13} \text{ s/s} \). With the above numbers we get \( I = 1.6 \times 10^{45} \text{ g cm}^2 \). The energy loss rate is therefore \( dE/dt = I\Omega \dot{\Omega} = 7 \times 10^{38} \text{ erg s}^{-1} \). Therefore, \( p_0 \sin \alpha \approx 5 \times 10^{38} \) and \( B \approx 3 \times 10^{12} \text{ G} \), if we set \( \sin \alpha \approx 1 \).

There are from this two interesting observations. The first is of course the extremely high magnetic field at the surface. The second is the fact
that the amount of energy lost due to the spin-down is very close to the total amount of energy emitted in the Crab nebula. This is dominated by synchrotron emission in the X-rays, and this shows that the energy needed from this is likely to originate from the pulsar.

6.3 The magnetosphere. Longair p. 107-108

Because the magnetic field is rotating with the neutron star, there will be a radius for which the corotation velocity is equal to the speed of light. This is given by $R_{c.c.} = c/\Omega = 5 \times 10^4 P$ km. For radii larger than this the field lines would rotate faster than the speed of light. Field lines outside this radius must therefore lag behind rigid rotation, and get a toroidal component.

Figure 11: Magnetic field structure around the neutron star. (Lorimer & Kramer)
The dipole field is given by Eq. (6.8). It is not difficult to show that the magnetic field lines in this case are given by

\[ r = K \sin^2 \theta \]

(6.9)

where \( K \) is the parameter which defines the line. Of special interest is the last field line which is closed within the light cylinder. At the surface of the neutron star the azimuthal angle of this line is given by

\[ \sin \theta_{l.c.} = \left( \frac{R \Omega}{c} \right)^{1/2}. \]

(6.10)

For the parameters of the Crab pulsar \( \theta_{l.c.} \sim 5 \text{ deg} \). For \( \theta > \theta_{l.c.} \) the field lines are closed, while for \( \theta < \theta_{l.c.} \) they are open.

The extremely rapidly rotating magnet will induce a very strong Lorentz force just outside the surface, \( F \sim e v x B \approx e \Omega R B \). Compared to the gravitational this is for an electron \( F/F_G \approx e R^3 B / G M m_e \approx 10^{12} \). Therefore even if there was a vacuum outside of the pulsar, the electric force would be so strong that electrons and protons would escape from the surface and flow into the magnetosphere. The region outside the neutron star will therefore become filled with charged particles which will neutralize the electric field here. The electric force will mainly be parallel to the magnetic field lines. Particles escaping on field lines which close will mainly accumulate within this magnetosphere, while those coming from the poles on open field lines will escape to infinity, and therefore in this way transport energy to the surrounding medium.
7 Neutron stars. Longair 15.3.3

The Chandrasekhar mass does not involve the mass of the particle responsible for the degenerate pressure. The maximum mass of a star made of neutrons should therefore be similar. The structure of the star should, however, be much more compact, as can be seen from the mass–radius relation, Eq. (4.8), which shows that the radius should be a factor $m_e/m_n \approx 1800$ smaller. Except for this we do, however, expect a similar relation between mass and radius.

At the next level of approximations there are, however, two effects which change these conclusions. The first is that the equation of state is not that of a pure non-interacting neutron gas. Instead nuclear interacting and other effects become important as the density is close or above nuclear.

The second effect comes from the fact that the radius of a neutron star, $\sim 10$ km, is of the same order as the Schwarzschild radius, $\sim 4$ km. General relativistic effects are therefore important for the structure of neutron stars. In this case the hydrostatic equation, Eq. (2.3), is replaced by the GR analogue which is known as the Oppenheimer-Volkoff equation

$$\frac{dP}{dr} = \frac{-G(m(r) + 4\pi r^3 \rho/c^2)(\rho + p/c^2)}{r(r - 2Gm(r)/c^2)}$$

The mass conservation equation is the same as before. Compared to the Newtonian case the pressure here adds to the mass as a source to the gravitational force. In addition, the curvature of space changes the $r^2$ term into $r(r - 2Gm(r)/c^2)$. All these corrections tend to increase the effect of gravity and therefore leads to a smaller mass compared to the Newtonian case. The neutron star analogue to the Chandrasekhar mass for a pure gas of neutrons, the Oppenheimer-Volkoff mass, is therefore 0.71 $M_\odot$ corresponding to a radius 9.14 km. In contrast to the white dwarf case the pure neutron star EOS is, however, not a very good approximation and more realistic cases result in considerably higher masses, as we will see below. The uncertainties in the structure and the EOS at especially densities larger than nuclear are unfortunately large. We will now discuss the structure in more detail, starting from the surface.

7.1 Crust

Close to the surface there is an ‘atmosphere’ where the conditions very fast go from non-degenerate to that of a white dwarf EOS (equation of state). The thickness of this region is only $\sim 1$ cm.
The crust extends to a density of $\sim 10^{14}$ g cm$^{-3}$, and the thickness of the crust is $\sim 1$ km, although this may vary by a factor of about two between different models.

In the interior the density quickly increases, and with that the Fermi energy, $E_F = (p_F^2 - m_e^2c^2)^{1/2}$ where $p_F = (3h^3n_e/8\pi)^{1/3}$ (Eq. (3.4)). For densities $\gtrsim 10^6$ g cm$^{-3}$ this energy may become large enough to overcome the mass difference between the proton and neutron, converting protons in the nuclei to neutrons, making the matter increasingly neutron rich, referred to as neutronisation. The matter is therefore changing from ordinary $^{56}$Fe to extremely neutron rich nuclei with $A \sim 200$ and $Z/A \sim 0.1$. At this point the neutron starts to drip out of the nuclei, forming a neutron gas together with the degenerate electrons and nuclei. This is usually referred to as the neutron drip and occurs at $\sim 4 \times 10^{11}$ g cm$^{-3}$. Most of the neutrons are, however, contained within the nuclei until a density of $\sim 10^{14}$ g cm$^{-3}$.

The nuclei are located in a lattice whose structure changes from that of an individual nuclei to distinctly non-spherical (from ‘meatballs, to spaghetti, to lasagna, to o Swiss cheese’). At $\sim 10^{14}$ g cm$^{-3}$ the nuclei begin to break up and form a gas of free neutrons together with degenerate electrons. This occurs at about half nuclear density, or $\sim 10^{14}$ g cm$^{-3}$.

Because pairing of free neutrons give a lower energy state the neutrons will probably form a superfluid.

### 7.2 Core

Nuclear density corresponds to $\rho_n \sim 2.8 \times 10^{14}$ g cm$^{-3}$.

The core region contains $\sim 99\%$ of the mass of the NS. The density in the core is rather constant, varying by only a factor of $\sim 2$. The outer core with density $\lesssim 2\rho_n$ is reasonably well understood, and consists of neutrons with a few percent of protons and electrons, and possibly muons. All are degenerate. Neutrons and protons form a superfluid.

The extent of the inner core depends sensitive on the EOS and the mass of the star. Superfluidity of protons also means that the protons are superconducting. At very high densities, well above $\rho_n$, the pairing may, however, be suppressed.

The composition of the inner core is to a large extent open. There are several possibilities discussed. The most conservative is that it has the same composition as the outer core, i.e., neutrons with a few percent of protons, electrons and muons. An often discussed possibility is that a Bose-Einstein condensate of pions may form. A variation of this is a condensate of $K$-mesons (kaons). Finally, at high enough density the nucleons may merge.
The main effect of these different possibilities for the EOS is that the EOS will be more or less stiff. The creation of pion and kaon condensates, as well as quarks, will in general make the EOS soft. A soft EOS has in turn the consequence that the matter can be compressed more, making the radius of the NS smaller. A measurement of the radius would therefore make it possible to test these different possibilities.

The radius can be determined from e.g. the thermal emission, if such a component can be observed (see below). A related effect is that the cooling of the neutron star is affected by the composition, and therefore the EOS in the inner core.

Shortly after the collapse and bounce the temperature in the core is $\sim 5 \times 10^{11} \text{ K}$ at 15 s, decreasing to $\sim 5 \times 10^{9} \text{ K}$ at 50 s. Most of the cooling then occurs by emission of neutrinos, which in turn depends on the composition. The most effective is the so called URCA process, named after
a casino in Rio de Janeiro by Gamow,

\begin{align*}
n \rightarrow p + e^- + \bar{\nu}_e \\
p + e^- \rightarrow n + \nu_e
\end{align*}

(7.2)

These two processes give no net change of the number of neutrons or protons, but produces two neutrinos, which carry away energy. The fact that this process always results in a loss of energy is the reason for its name. For this process to occur the proton concentration must be sufficiently high, \( \gtrsim 10\% \). This requires a high density and therefore a soft EOS, and is therefore a diagnostic of the EOS. In this case a modified URCA process may still occur

\begin{align*}
n + (n, p) & \rightarrow p + (n, p) + e^- + \bar{\nu}_e \\
p + (n, p) & \rightarrow n + (n, p) + e^+ + \nu_e
\end{align*}

(7.3)

where \((n, p)\) is an additional nucleon, neutron or proton, which ensures momentum conservation. This process can occur under more general conditions, but is much slower than the direct URCA process. The cooling therefore takes longer time and the interior temperature will be higher. After \( \sim 100 \) years the temperature will be \( \sim 3 \times 10^8 \) K in the former case and \( \sim 1.2 \times 10^9 \) K in the latter. The interior is now isothermal. From 100 years to \( \sim 3 \times 10^5 \) years the temperature is \( \sim 2 \times 10^8 \) K in the URCA case and \( \sim 6 \times 10^8 \) K in the modified URCA case.

The connection between the interior temperature and the surface temperature depends on the heat conduction. This is in turn dependent on magnetic fields and composition. The typical surface temperatures are in the range \( 3 \times 10^5 \) to \( 10^6 \) K. Figure 13 shows the expected temperatures for different assumptions about the cooling, together with observations from several NSs. It should, however, be pointed out that these determinations are difficult because for most pulsars the thermal emission from the surface is dominated by the non-thermal radiation connected with the pulsar mechanism in the magnetosphere. Most of the emission comes in the UV and in soft X-rays. The observations indicate that cooling occurs mainly by the modified URCA process, and that the direct is suppressed.

Another effect of the different EOS in the core is the maximum mass of the NS. A soft EOS gives in general a lower maximum NS mass, while conversely a stiff EOS results in a high. Fig. 14 shows the radius versus mass for different EOSs. The maximum mass for the different EOSs ranges from \( \sim 1.4 \, M_\odot \) to \( \sim 2.7 \, M_\odot \). A very firm upper bound comes from the requirement that the EOS should be such that the speed of sound is less
than the velocity of light. This mass is $\sim 3.0 \, M_\odot$. A determination of a larger mass for a compact object is therefore extremely strong evidence for the presence of a black hole (see below).

The masses of a large number of NSs in binary systems have been determined with different degree of accuracy. The most accurate are those from NSs in neutron star – neutron star systems, including the famous PSR 1913+16 (the Hulse and Taylor binary pulsar), as well as PSR 1534+12. In these cases the masses are $1.4408 \pm 0.0003 \, M_\odot$ and $1.3873 \pm 0.0003 \, M_\odot$ (PSR 1913+16) and $1.3332 \pm 0.0010 \, M_\odot$ and $1.3452 \pm 0.0010 \, M_\odot$ (PSR 1534+12). In Fig. 15 we show a compilation of masses for most of the observed neutron star binaries. These are determined either from timing of the pulsar frequency in different binaries containing at least one pulsar, or from the motion of the companion star in the case of X-ray binaries. The most accurate of these methods is obviously those systems containing two neutron stars, which is a very clean’ case of two point masses. As we see the
maximum mass of the most accurately determined are $\sim 1.44 \, M_\odot$. There are, however, several systems which are compatible with considerably higher masses and neither the stiff EOSs (with large masses) or the soft (with low masses) can unfortunately be ruled out yet.

It can be noted that there is a tendency for somewhat higher masses for neutron stars with white dwarf companions.
Figure 15: Compilation of neutron star masses determined from different types of neutron star binaries. The vertical dashed line shows a rough average mass of 1.38 M⊙ (data from Lattimer and Prakash)
7.3 Binary pulsars

The first discovered pulsar was PSR 1913+16, discovered by Hulse and Taylor. The pulsar period of the is 22.7 millisecond and the orbital period 7.75 hours. The eccentricity of the orbit is very high, \( e = 0.617 \), which is important for both the advance of the periastron and the decay by gravitational radiation.

The Post-Newtonian parameters are given by

\[
\dot{\omega} = 3T_{\odot}^{2/3} \left( \frac{P_b}{2\pi} \right)^{-5/3} \frac{1}{1-e^2} (M_p + M_c)^{2/3}, \quad (7.4)
\]

\[
\gamma = T_{\odot}^{2/3} \left( \frac{P_b}{2\pi} \right)^{1/3} \frac{M_c (M_p + 2M_c)}{(M_p + M_c)^{4/3}}, \quad (7.5)
\]

\[
\dot{P}_b = -\frac{192\pi}{5} T_{\odot}^{5/3} \left( \frac{P_b}{2\pi} \right)^{-5/3} \frac{(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4)}{(1 - e^2)^{7/2}} \frac{M_p M_c}{(M_p + M_c)^{1/3}}, \quad (7.6)
\]

\[
r = T_{\odot} M_c, \quad (7.7)
\]

\[
s = T_{\odot}^{-1/3} \left( \frac{P_b}{2\pi} \right)^{-2/3} x \frac{(M_p + M_c)^{2/3}}{M_c}, \quad (7.8)
\]

where \( P_b \) is the period and \( e \) the eccentricity of the binary orbit. The masses \( M_p \) and \( M_c \) of pulsar and companion, respectively, are expressed in solar masses \( (M_\odot) \). The constant \( T_{\odot} = GM_\odot/c^3 = 4.925490947 \mu s \). The first PN parameter, \( \dot{\omega} \), describes the relativistic advance of periastron. According to Eq. (7.4) it gives a direct measurement of the total mass of the system, \((M_p + M_c)\). The parameter \( \gamma \) denotes the amplitude of delays in arrival times caused by the varying effects of the gravitational redshift and time dilation (second order Doppler) as the pulsar moves in its elliptical orbit at varying distances from the companion and with varying speeds. The decay of the orbit due to gravitational wave damping is expressed by the change in orbital period, \( \dot{P}_b \). The other two parameters, \( r \) and \( s \), are related to the Shapiro delay caused by the gravitational field of the companion. These parameters are only measurable, depending on timing precision, if the orbit is seen nearly edge-on.

For PSR 1913+16 the gravitational decay parameter is \( \dot{P}_b = -2.4184 \times 10^{-12} \) s/s (Fig. reffigpsr193). Based on the gravitational decay the system should merge in \( \sim 3 \times 10^8 \) years. The periastron advance is given by \( \dot{\omega} = 4.22 \) degrees/yr. The most recent determination of the individual pulsar masses are \( 1.4414 \pm 0.0002 \) M\(_\odot\) and \( 1.3867 \pm 0.0002 \) M\(_\odot\). Besides PSR 1913+16, the binary PSR B1534+14 has provided a similar test of relativity.
The most interesting of the binary pulsars discovered after PSR 1913+16 is the double-pulsar PSR J0737-3039. This is unique in the sense that two pulsars are seen in this system, which gives additional constraints on the parameters of the system. The individual periods are 22.7 ms and 2.773 s. Another unique feature is that it is seen nearly edge-on with $\sin i = 0.9995$, i.e. $i = 87$ degrees. This makes it possible not only to study the orbital parameters, but also to probe the pulsar magnetospheres. The orbital period is 2.4 hours. The periastron advance $\dot{\Omega} = 16.88$ degrees, which is four times larger than for PSR 1913+16. The edge-on nature of the system has also provided a new observational constraint from the ratio of the masses of the system, $R = M_1/M_2$. The system will be coalescing within 85 million years, much shorter than for PSR 1913+16. This in connection to the low
luminosity has increased the frequency of merging neutron stars by an order of magnitude, which is important both for the detection of gravitational waves and for merging neutron stars as a source of the short gamma-ray bursts. In Fig. 17 we show the solution for the two masses as function of the different post-Newtonian parameter, including $R$. From it is found that the neutron star masses are $1.337 \pm 0.005 \, M_\odot$ and $1.250 \pm 0.005 \, M_\odot$.

![Figure 17: Orbital parameters in the $M_1, M_2$ plane for Shift of periastron for PSR J0737-3039.](image)
7.4 The binary mass function. (Longair p. 113)

Let \( a_1 \) and \( a_2 \) be the distance from the CM, i.e.
\[
a = a_1 + a_2 \tag{7.9}
\]
and
\[
M_1 a_1 - M_2 a_2 = 0 \tag{7.10}
\]
or
\[
a = \frac{(M_1 + M_2) a_1}{M_2} \tag{7.11}
\]

We observe the projected orbital velocity \( v_1 \), given by
\[
v_1 = \frac{2\pi}{P} = a_1 \sin i \tag{7.12}
\]
where \( i \) is the inclination. Kepler’s law says that
\[
\frac{G(M_1 + M_2)}{a^3} = \left(\frac{2\pi}{P}\right)^2 \tag{7.13}
\]
Therefore
\[
f(M_1, M_2, i) = \frac{(M_2 \sin i)^3}{(M_1 + M_2)} = \frac{P v_1^3}{2\pi G} \tag{7.14}
\]
We can observe the RHS of this equation. As shown by this equation, this, however, only gives the mass function. In addition we need to know the inclination and the mass ratio.

8 Hydrodynamics

8.1 The equations of fluid dynamics

Consider a volume, \( V \), surrounded by a surface, \( S \), with a flux of particles flowing in and out of the volume through the surface. The mass flowing out through a surface element \( dS \) (with normal out from the volume) per unit time is \( \rho v dS \). This flow integrated over the whole surface is of course the same as the decrease in mass in the volume, or
\[
- \int \frac{\partial \rho}{\partial t} dV = \int \rho v dS \tag{8.1}
\]

With Gauss theorem the surface integral can be converted to a volume integral,
\[
- \int \frac{\partial \rho}{\partial t} dV = \int \nabla \cdot (\rho v) dV \tag{8.2}
\]
Because this should be true for any volume we must have

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot \rho \mathbf{v} \tag{8.3}
\]

which is the continuity equation for the mass.

Next let us consider the force acting on the surface through the pressure, \(- pd\mathbf{S}\) (note sign). Again using Gauss theorem we have

\[
- \int p d\mathbf{S} = - \int \nabla p dV \tag{8.4}
\]

The force on \(dV\) as it moves around is therefore \(- \nabla p dV\), so that

\[
\rho \frac{d\mathbf{v}}{dt} = - \nabla p + \rho \nabla \phi \tag{8.5}
\]
where $\phi$ is the gravitational potential. This is the Lagrangian (comoving) derivative, so in Eulerian coordinates we get

$$
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \rho \nabla \phi \tag{8.6}
$$

This is the Euler equation.

Euler’s equation can also be written as a conservation law of momentum per unit volume, $\rho \mathbf{v}_i$, similar to that of mass. Using the mass and momentum conservation above we have

$$
\frac{\partial \rho \mathbf{v}_i}{\partial t} = -\rho \sum_j v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho \frac{\partial \phi}{\partial x_i} - \mathbf{v}_i \sum_j \frac{\partial \rho v_j}{\partial x_j} = -\sum_j \frac{\partial \rho v_j v_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho \frac{\partial \phi}{\partial x_i} \tag{8.7}
$$

If we write

$$
T_{ij} = p \delta_{ij} + \rho v_i v_j \tag{8.8}
$$

where $\delta_{ij}$ is 0 for $i \neq j$ and 1 for $i = j$, this can be written as

$$
\frac{\partial \rho \mathbf{v}_i}{\partial t} = -\sum_j \frac{\partial T_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i} \tag{8.9}
$$

The tensor $T_{ij}$ gives the momentum flux of the $j$ momentum component in direction $i$, including both the mass flux and the pressure. It is usually called the energy-momentum tensor.

Finally we consider the energy of the volume. This is the sum of the kinetic and internal energy $\rho \mathbf{v}^2/2 + \rho \epsilon$. Here, $\epsilon$ is the internal energy per unit mass. Let us consider the evolution of this quantity with time,

$$
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \epsilon \right) = \frac{\mathbf{v}^2}{2} \frac{\partial \rho}{\partial t} + \sum_i \rho v_i \frac{\partial v_i}{\partial t} + \epsilon \frac{\partial \rho}{\partial t} + \rho \frac{\partial \epsilon}{\partial t} \tag{8.10}
$$

which clearly has the form of a conservation law, and is analogous to the mass conservation law Eq. (8.6).

The connection between internal energy, pressure, volume and heat loss $(Tds)$ is given by

$$
d\epsilon = -p dV + T ds = -p dV + T ds \tag{8.11}
$$

where $V$ is the specific volume (i.e., volume per unit mass), so $V = 1/\rho$ and $s$ is the entropy, also per unit mass.
Using this together with mass conservation and momentum conservation Eqs. (8.3) and (8.6) and writing the heat loss term as

$$\rho T \frac{ds}{dt} + \sum_i v_i \frac{\partial s}{\partial x_i} = \rho T \frac{ds}{dt} = \Lambda$$

one can now transform Eq. (8.10) to

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho \epsilon \right) = -\sum_j \frac{\partial}{\partial x_j} \left[ \rho v_j \left( \frac{v^2}{2} + w \right) \right] + v_j \frac{\partial \phi}{\partial x_j} + \Lambda$$

where $w = \epsilon + p/\rho = \gamma/(\gamma - 1) p/\rho$ is the heat function. This is equivalent to

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho \epsilon \right) = -\nabla \cdot \left[ \rho v \left( \frac{1}{2} \rho v^2 + \rho \epsilon + p \right) \right] + \rho \nabla \cdot \phi + \Lambda$$

This equation says that the change in total energy density in a volume is equal to the flux of kinetic and internal energy through its surface (the $\rho v (v^2/2 + \epsilon)$ term) plus the work done on this volume by the pressure (the $v p$ term), plus the heat lost by other processes, like conduction or radiation.

Equations (8.3), (8.6) and (8.14) (or Eq. (8.13)) constitute the complete set of hydrodynamic equations.

A case which often is important is that of a stationary, spherically symmetric flow. In this case all $\partial/\partial t$-terms, as well as angular derivatives are zero. Further, $\nabla \cdot f = r^{-2} \partial (r^2 f_r)/\partial r$ (since $\partial/\partial \theta = \partial/\partial \phi = 0$), and we get

$$0 = -\frac{1}{r^2} \frac{\partial}{\partial r} \left[ v_r \left( \frac{1}{2} \rho v^2 + \rho \epsilon + p \right) \right] + \frac{\partial \phi}{\partial r} + \Lambda$$

Finally we note that Eq. (8.6) can be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \rho \nabla \cdot v = \frac{d\rho}{dt} + \rho \nabla \cdot v = 0$$

For terrestrial applications it is often a good approximation to consider the fluid as *incompressible*, i.e. that the density does not change as the fluid element moves. This means that $d\rho/dt = 0$, or from the equation above that $\nabla \cdot v = 0$.

### 8.2 The shock jump conditions

The speed of sound for an adiabatic gas is given by

$$c_s^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho}$$

34
where $\gamma$ is the adiabatic index. For a non-relativistic gas $\gamma = 5/3$ and for a relativistic $\gamma = 4/3$.

Consider a wave propagating with the sound velocity, initially with a sinusoidal form (see Fig. 19). The crest of the wave will have a density slightly higher than the trough. Because $c_s^2 \propto p/\rho \propto \rho^{\gamma-1}$ the sound velocity will therefore be higher in the crest than in the through, which means that the crest will propagate faster than the through. The initially sinusoidal wave will therefore gradually become steeper, and at some point the crest will catch up with the through. The wave will then break, as is familiar from the sea. Mathematically this is the same as a discontinuity.

![Figure 19: Non-linear development of a wave.](image)

Before the breaking of the wave the motion of the particles in the fluid could be considered as dissipation less, i.e., the viscosity played little role. One the wave brakes the particles collide and the viscosity will lead to dis-
sipation, resulting in a non-adiabatic process.

An even more dramatic situation will occur if some object, e.g., an airplane, is moving with a velocity higher than that of the sound. For objects with velocity below that of the sound, sound waves will propagate ahead of it, which will affect the density and pressure of the medium. The medium is therefore ‘prepared’ for the arrival of the object, and a fairly smooth transition takes place. This is not possible for a supersonic object. Here, the sound waves cannot send this kind of early warning to the system, since they constantly lag behind the object. The arrival of the object will therefore only affect the medium once it has already arrived. The transition from the undisturbed medium to the disturbed will therefore take place very suddenly, and will for a thin transition region where the atoms of swept up by the object will collide with those of the undisturbed medium. Here viscosity will play a very important role and as above a dissipative process with a large change in entropy will take place. The thickness of the transition will be of the order of the mean free path of the particles. For most purposes this shock wave can be considered as a mathematical discontinuity, although for certain processes like non-thermal particle acceleration or non-thermal heating the actual structure is important.

To describe the relation between the conditions ahead and behind the shock wave we use the conservation laws derived in the previous section.

Let us therefore consider a very thin region of the flow with the preshock gas on one side. Further, we make a Galilean transformation into the reference system of the shock. All velocities will therefore from now on refer to that of the shock discontinuity. If e.g., the shock moves with velocity \( v_s \) relative to an observer at rest relative to the gas into which the shock propagates, the velocity of the gas coming in to the shock will be \( v_s \). The density, velocity and pressure are here denoted by \( \rho_1, v_1, p_1 \), while those behind the shock are denoted by \( \rho_2, v_2, p_2 \). Because the region is very thin the flow can be considered one-dimensional, and time independent.

The mass conservation equation (8.3) then become

\[
\frac{d}{dx} \rho v = 0 \tag{8.18}
\]

Therefore we have

\[
\rho_2 v_2 = \rho_1 v_1 = J \tag{8.19}
\]

The momentum equation, Eq. (8.6), becomes

\[
\rho v \frac{dv}{dx} + \frac{dp}{dx} = 0 \tag{8.20}
\]
From Eq. (8.18) \( \rho v = \text{const.} \), so \( \rho v^2 + p = \text{constant} \), and

\[
\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{8.21}
\]

Finally, the energy equation, Eq. (8.14), becomes

\[
\frac{d}{dx} \left[ \rho v \left( \frac{v^2}{2} + w \right) \right] = 0 \tag{8.22}
\]

Again using \( \rho v = \text{const.} \) we get

\[
\frac{v_1^2}{2} + w_1 = \frac{v_2^2}{2} + w_2 \tag{8.23}
\]

Let us now specialize this to a perfect gas where \( \epsilon = (\gamma - 1)^{-1} p/\rho \) The heat function is \( w = \epsilon + p/\rho = \gamma/(\gamma - 1) \frac{p}{\rho} \). Therefore Eq. (8.23) becomes

\[
\frac{v_1^2}{2} + \frac{p_1}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{v_2^2}{2} + \frac{p_2}{\gamma - 1} \frac{p_2}{\rho_2} \tag{8.24}
\]

Let us now further specialize this to the case when the pressure of the gas in front of the shock is much smaller than that in the post shock gas, i.e., \( p_1 \ll p_2 \). This is known as the strong shock condition. The momentum condition Eq. (8.21) then becomes

\[
\rho_1 v_1^2 = \rho_2 v_2^2 + p_2 \tag{8.25}
\]

and the energy condition Eq. (8.24)

\[
\frac{v_1^2}{2} = \frac{v_2^2}{2} + \frac{p_2}{\gamma - 1} \frac{p_2}{\rho_2} \tag{8.26}
\]

Using Eqns. (8.19), (8.25) and (8.26) one then finds that

\[
\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma - 1)}{(\gamma + 1)} \tag{8.27}
\]

One then finds that

\[
p_2 = \frac{2}{(\gamma + 1)} \rho_1 v_1^2 \tag{8.28}
\]

and using \( p = k\rho T/\mu m_p \), where \( \mu \) is the molecular weight and \( m_p \) the atomic mass unit, we get

\[
T_2 = \frac{2(\gamma - 1) \mu m_p}{k(\gamma + 1)^2} v_1^2 \tag{8.29}
\]
For \( \gamma = 5/3 \) we have
\[
\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{1}{4}
\] (8.30)

One then finds that
\[
p_2 = \frac{3}{4} \rho_1 v_1^2
\] (8.31)

and
\[
T_2 = \frac{3}{16} \frac{\mu m_p}{k} v_1^2
\] (8.32)

The mean molecular weight is for a fully ionized mix of hydrogen and helium
\[
\mu = \left( \frac{n(H) + 4n(He)}{2n(H) + 3n(He)} \right)
\] For \( n(He)/n(H) \approx 0.1 \) we get
\[
\mu = 0.61 \text{ and } T_2 = 1.38 \times 10^7 \left( \frac{v_1}{1000 \text{ km s}^{-1}} \right)^2 \text{ K.}
\] (8.33)

9 Viscosity

The momentum transfer connected to the particle flow and pressure as given by Euler’s equation Eq. (8.6), only involves reversible processes. There may, however, also be processes involving irreversible, dissipative processes. This is known as viscous processes, and are familiar from fluids like water, oil etc. Also gases are characterized by a viscosity. For these the viscosity is connected with molecular processes, There may, however, also be macroscopic processes which can have the same effect. An example is turbulence, which is important in e.g., accretion disks.

To describe the viscous momentum transfer we should add a term to the Euler equation. This means an additional term to the energy momentum tensor in Eq. (8.8). This term should obviously depend on the derivative of the velocity. If we have a flow in the x-direction, the momentum transfer connected with the friction should be proportional to the derivative of the velocity in the y-direction,
\[
\tau_{xy} = a \frac{\partial v_x}{\partial y}
\] (9.1)

For a fluid in circular motion the friction should depend on both \( \partial v_x/\partial y \) and \( \partial v_y/\partial x \). In addition for a fluid with angular velocity independent of the distance from the center (solid body rotation) the friction should vanish. For this \( \mathbf{v} = \Omega \mathbf{r} \), where \( \Omega \) is independent of \( \mathbf{r} \). Therefore \( v_x = \Omega y \) and \( v_y = -\Omega x \). The only combination of the above derivatives which gives zero friction is
\[
\tau_{xy} = a(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}) = \tau_{yx}
\] (9.2)
In addition to these shear stresses there may also be dissipation due to compression in either direction. This should be proportional to $d\rho/dt$ or $\nabla \cdot \mathbf{v}$ (see Eq. (8.16)),

$$\tau_{xx} = b\nabla \cdot \mathbf{v} = \tau_{yy}$$  \hspace{1cm} (9.3)

The most general expression generalized to three dimensions is therefore

$$\tau_{ij} = a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + b\delta_{ij}\sum_k \frac{\partial v_k}{\partial x_k}$$  \hspace{1cm} (9.4)

Usually one writes this as a traceless part and a diagonal

$$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\sum_k \frac{\partial v_k}{\partial x_k}\right) + \zeta \delta_{ij}\sum_k \frac{\partial v_k}{\partial x_k}$$  \hspace{1cm} (9.5)

Here $\eta$ is called the shear viscosity and $\zeta$ the bulk viscosity. For an incompressible fluid only the former is important. This should be added to the perfect gas expression for the energy-momentum tensor in Eq. (8.8).

$$T_{ij} = p\delta_{ij} + \rho v_i v_j + \tau_{ij}$$  \hspace{1cm} (9.6)

### 9.1 The Navier-Stokes equation

Adding the viscosity tensor to the energy-momentum tensor in Eq.(8.9) we get the equation of motion, including viscosity

$$\frac{\partial \rho v_i}{\partial t} = -\sum_j \frac{\partial T_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i} = -\frac{\partial p}{\partial x_i} - \sum_j \frac{\partial \rho v_i v_j}{\partial x_j} - \sum_j \frac{\partial \tau_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i}$$  \hspace{1cm} (9.7)

To simplify the writing we will from now on use the Einstein’s sum convention. This says that whenever an index occurs twice this implicitly means a summation over this index. E.g., $\sum_j A_j B_i \equiv A_i B_i$ and

$$\nabla \cdot \mathbf{v} = \sum_i \frac{\partial v_i}{\partial x_i} \equiv \frac{\partial v_i}{\partial x_i}$$

The equation above is therefore equivalent to

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial T_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i} = -\frac{\partial p}{\partial x_i} - \frac{\partial \rho v_i v_j}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i}$$  \hspace{1cm} (9.8)

If we assume that the viscosity coefficients $\eta$ and $\zeta$ are independent of temperature and pressure, i.e., of $x_i$, we get

$$\frac{\partial \tau_{ij}}{\partial x_j} = \eta \frac{\partial^2 v_i}{\partial x_i \partial x_j} + \left(\zeta + \frac{2}{3}\eta\right) \frac{\partial^2 v_j}{\partial x_i \partial x_j}$$  \hspace{1cm} (9.9)
(note that $\delta_{ij} \partial/\partial x_j = \partial/\partial x_i$). Using the mass continuity equation we can now write Eq. (9.7) as

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} - (\zeta + \frac{2}{3} \eta) \frac{\partial^2 v_j}{\partial x_i \partial x_j} + \rho \frac{\partial \phi}{\partial x_i} \quad (9.10)$$

or in vector notation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \eta \nabla \Delta \mathbf{v} - (\zeta + \frac{2}{3} \eta) \nabla \nabla \cdot \mathbf{v} + \rho \nabla \phi \quad (9.11)$$

where $\Delta$ is the Laplace operator, $\Delta = \sum_i \partial^2 / \partial x_i^2$. This is the Navier-Stokes equation for a viscous fluid.

In the case of an incompressible fluid, $\nabla \cdot \mathbf{v} = 0$, the momentum tensor due to viscosity, Eq. (9.5), and the Navier-Stokes equation above take a simpler form,

$$\tau_{ij} = \eta (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \quad (9.12)$$

and

$$\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \eta \nabla \Delta \mathbf{v} + \rho \nabla \phi \quad (9.13)$$

### 9.2 The energy equation for a viscous fluid

For a fluid with no viscosity the energy conservation equation Eq. (8.14) is

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \epsilon \right) = -\nabla \cdot [\mathbf{v} (\frac{1}{2} \rho \mathbf{v}^2 + \rho \epsilon + p) + \rho \mathbf{v} \cdot \nabla \phi \nabla \phi + \Lambda] \quad (9.14)$$

which says that the change of total energy in a volume (LHS) is equal to the energy flux across its boundary (RHS).

Also the work done by the viscous force generates an energy flux. To see this we note that $\tau_{ij} dS_j$ is the viscous force in the direction $i$ on the surface $j$. The work done per unit time on the volume in direction $i$ is therefore $v_i \tau_{ij} dS_j$. (Because we are here considering the individual contributions for each $i$ and $j$, there is here no summation implied.) The total change in energy is therefore

$$\frac{dE}{dt} = \int v_i \tau_{ij} dS_i \quad (9.15)$$

where as usual a sum over $i$ and $j$ is implied, since we now consider the total contribution from all directions and on all sides. Applying Gauss theorem to the surface integral we get

$$\frac{dE}{dt} = \int \frac{\partial}{\partial x_j} (v_i \tau_{ij}) dV \quad (9.16)$$
Therefore, we should add this term to Eq. (9.14) giving in component form

\[
\frac{\partial}{\partial t}\left( \frac{1}{2} \rho \dot{v}_i^2 + \rho \epsilon \right) = -\frac{\partial}{\partial x_i}[v_i(\frac{1}{2} \rho \dot{v}_j^2 + \rho w) + v_j \tau_{ij}] + \rho v_i \frac{\partial \phi}{\partial x_i} + \Lambda \tag{9.17}
\]

where \( w = \epsilon + p/\rho \) is the heat function. This is the full energy equation including viscosity.

To more clearly see the effect of the viscous heating we calculate the total energy dissipation \( \rho T ds/dt \) per volume and unit time as the fluid element moves. Transforming from the comoving, Lagrangian system to a fixed Eulerian we get

\[
\rho T \frac{ds}{dt} = \rho T \left[ \frac{\partial s}{\partial t} + v_i \frac{\partial s}{\partial x_i} \right] \tag{9.18}
\]

To rewrite this equation we use the law of thermodynamics, \( ds = -pdV + Tds = p/\rho^2 d\rho + Tds \) to eliminate \( \partial s/\partial t \). In the same way we use \( dw = TdS + dp/\rho \) to eliminate \( \partial s/\partial x_i \) to get

\[
\rho T \frac{ds}{dt} = \rho T \left[ \frac{\partial \epsilon}{\partial t} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial t} + v_i \frac{\partial w}{\partial x_i} - \frac{v_i}{\rho} \frac{\partial p}{\partial x_i} \right] \tag{9.19}
\]

We now use the energy conservation equation Eq. (9.17) to calculate the term \( \rho (\partial \epsilon/\partial t + v_i \partial w/\partial x_i) \). We neglect for the moment the gravitational potential. Using mass conservation we find from Eq. (9.17)

\[
\rho \left( \frac{\partial \epsilon}{\partial t} + v_i \frac{\partial w}{\partial x_i} \right) = -\rho v_i \frac{\partial v_i}{\partial t} - \frac{p}{\rho} \frac{\partial \rho}{\partial x_i} - \rho v_i v_j \frac{\partial v_j}{\partial x_i} + \frac{\partial \rho \tau_{ij}}{\partial x_j} \tag{9.20}
\]

We now use the Navier-Stokes equation

\[
\rho \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \rho \frac{\partial \phi}{\partial x_i} \tag{9.21}
\]

multiplied by \( v_i \) to eliminate the \( \rho v_i \partial v_i/\partial t \) term in Eq. (9.20), resulting in

\[
\rho \left( \frac{\partial \epsilon}{\partial t} + v_i \frac{\partial w}{\partial x_i} \right) = v_i \frac{\partial p}{\partial x_i} - \frac{p}{\rho} \frac{\partial \rho v_i}{\partial x_i} + \frac{\partial \rho \tau_{ij}}{\partial x_j} \tag{9.22}
\]

Using this in Eq. (9.19) we finally get

\[
\rho T \frac{ds}{dt} = \tau_{ij} \frac{\partial v_i}{\partial x_j} \tag{9.23}
\]

for the energy dissipation due to the viscous forces per volume and unit time. If there are radiation losses this adds the term \( \Lambda \) (see Eq. (9.17).
In addition, heat conduction or other diffusive energy losses may also contribute. In particular, heat conduction is often important, either through Coulomb collisions or through radiation (see Eq. 2.10). The energy flux is for this proportional to the temperature gradient,

\[
F_i = -\kappa \frac{\partial T}{\partial x_i}
\]  

(9.24)

Note that the flux is opposite to the temperature gradient. The energy loss per volume is then (using again Gauss theorem)

\[
\frac{\partial F_i}{\partial x_i} = -\frac{\partial}{\partial x_i} \left( \kappa \frac{\partial T}{\partial x_i} \right)
\]  

(9.25)

which should be added to Eq. (9.23). In general the conductivity \( \kappa \) is a function of temperature.

The Navier-Stokes equation, Eq. (9.21), and Eq. (9.23) for the viscous energy dissipation will be useful especially in connection to the hydrodynamics of accretion disks.

In the case of an incompressible fluid Eq. (9.23) takes a simple form. Using Eq. (9.12) in this we get

\[
\rho T \frac{ds}{dt} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2
\]

(9.26)

which can also be written

\[
\rho T \frac{ds}{dt} = \frac{\eta}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2
\]

(9.27)

10 Spherical accretion

As an application of the hydrodynamical equations we consider the adiabatic accretion of gas from a constant density medium at rest onto a central gravitating body with mass \( M \). We therefore start with the time independent version of Eq. 8.14 with \( \phi = GM/r \)

\[
\frac{1}{2} v^2 + \left( \epsilon + p/\rho \right) - \frac{GM}{r} = \text{constant}
\]

(10.1)

or

\[
\frac{1}{2} v^2 + \frac{\gamma}{(\gamma - 1)} \frac{p}{\rho} - \frac{GM}{r} = \text{constant}
\]

(10.2)
But $c_s^2 = \gamma p/\rho$, so

$$\frac{1}{2}v^2 + \frac{c_s^2}{(\gamma - 1)} - \frac{GM}{r} = \text{constant} \quad (10.3)$$

The value of the constant depends on the problem we consider. In our case with accretion from a stationary medium with $v \approx 0$ as $r \to \infty$ we have constant $= c_s^2(\infty)/(\gamma - 1)$, so

$$\frac{1}{2}v^2 + \frac{c_s^2}{(\gamma - 1)} - \frac{GM}{r} = \frac{c_s^2(\infty)}{(\gamma - 1)} \quad (10.4)$$

In addition we have the mass conservation which says that

$$\dot{M} = 4\pi r^2 v \rho \quad (10.5)$$

where $\dot{M}$ is the accretion rate.

The relation above contains both $c_s(r)$ and $v(r)$. In principle we can use the relation $c_s^2 = \gamma p/\rho = \gamma K \rho^{\gamma - 1}$ and Eq. (10.5) to eliminate $c_s^2$ in Eq. (10.4). This, however, only gives $v$ as a function of $r$ and $\dot{M}$. The latter should, however, not be a free parameter, but should be determined from the values of the density and sound velocity at infinity.

To proceed further we go back to the momentum equation which for a stationary, spherically symmetric flow gives

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM}{r^2} \quad (10.6)$$

But $dp/dr = dp/d\rho \frac{d\rho}{dr} = c_s^2 d\rho/dr$. Eq. (10.5) shows that $d \ln \rho/dr = -d \ln vr^2$. Therefore

$$v \frac{dv}{dr} = c_s^2 \frac{d \ln vr^2}{dr} - \frac{GM}{r^2} \quad (10.7)$$

or

$$\left(1 - \frac{c_s^2}{v^2}\right) \frac{dv^2}{dr} = -\frac{2GM}{r^2} + \frac{4c_s^2}{r} \quad (10.8)$$

This equation is interesting because it has an obvious singularity for the RHS at

$$r_c = \frac{GM}{2c_s^2} \quad (10.9)$$

This means that also the LHS must be zero for $r = r_c$. There are two possibilities for this. Either $dv/dr = 0$ at $r_c$, or $v = c_s$ at $r_c$.

For our outer boundary condition the $dv/dr = 0$ possibility corresponds to solutions which accelerate to a maximum velocity and then decreases (see
Figure 20: Different solutions corresponding to different boundary conditions of the flow equations. Solution number 2 corresponds to the accretion onto a compact object. Solution number 1 corresponds to a wind, with an outer boundary condition with negligible pressure.

Fig. 20). This means that the density and pressure will increase rapidly for small \( r \). For this to be possible the accretion solution must match some kind of atmosphere. The flow will everywhere be subsonic.

The \( v = c_s \) at \( r_c \) solution means that the flow becomes supersonic at \( r_c \), which is the reason for sometimes naming it the sonic radius. This is, however, only correct for this type of solutions.

Using Eq. (10.9) in Eq. (10.4) with \( v = c_s \) at \( r = r_c \) we get

\[
\frac{1}{2} c_s^2 + \frac{c_s^2}{(\gamma - 1)} - 2c_s^2 = \frac{c_s^2(\infty)}{(\gamma - 1)}
\]  

(10.10)
or
\[
c_s(r_c) = c_s(\infty) \left( \frac{2}{5 - 3\gamma} \right)^{1/2}
\]  
(10.11)

We can now use this in Eq. (10.5) evaluated at \( r_c \)
\[
\dot{M} = 4\pi r_c^2 c_s(r_c) \rho(r_c) = \frac{\pi (GM)^2 \rho(r_c)}{c_s(r_c)^3}
\]  
(10.12)

For an adiabatic flow \( \rho(r_c) = \rho(\infty)(c_s(r_c)/c_s(\infty))^{(2/(\gamma - 1)), \text{ so}} \)
\[
\dot{M} = \frac{\pi (GM)^2 \rho(\infty)}{c_s(\infty)^3} \left( \frac{2}{5 - 3\gamma} \right)^{(3-3\gamma)/2(\gamma-1)}
\]  
(10.13)

This gives finally the accretion rate as function of the density and sound velocity at infinity. This is usually referred to as the Bondi accretion rate. The sound velocity is given by
\[
c_s = \left( \frac{\gamma}{\rho} \right)^{1/2} = \left( \frac{qk \mu T}{m_p} \right)^{1/2} = \left( \frac{\gamma kT}{\mu m_p} \right)^{1/2} = 12 \left( \frac{T}{10^4 \text{K}} \right)^{1/2} \text{ km s}^{-1}
\]  
(10.14)

for \( \gamma = 1 \) For the ISM we have \( c_s \approx 10 \text{ km s}^{-1} \) and \( \rho \approx 1.6 \times 10^{-24} \text{ g cm}^{-3} \) (i.e., one atom per cm\(^{-3}\)). This gives for \( \gamma \approx 1.4 \)
\[
\dot{M} \sim 10^{11} \left( \frac{M}{1 \, \text{M}_\odot} \right)^2 \frac{\rho(\infty)}{1.6 \times 10^{-24} \text{ g cm}^{-3}} \left( \frac{c_s(\infty)}{10 \text{ km s}^{-1}} \right)^3 \text{ g s}^{-1}
\]  
(10.15)

This is a very small accretion rate (\( \sim 10^{-15} \text{ M}_\odot \text{ yr}^{-1} \)) and accretion from the ISM is usually unimportant.

For \( r \ll r_c \) the gravitational attraction dominates over the pressure term in Eq. (10.4) and the gas is in free fall,
\[
v = \left( \frac{2GM}{r} \right)^{1/2}
\]  
(10.16)

The density is then
\[
\rho = \frac{\dot{M}}{4\pi(2GM)^{1/2}r^{3/2}}
\]  
(10.17)

We also remark that Eq. (10.3) also applies for an outflow, like the solar wind, but now with a different constant and a solution which starts subsonic at small radii, and ends with a supersonic flow at large radii.
11 Standard disks

The disk equations in Longair needs to be supplied by an equation of state. For a non-degenerate gas we have

\[ P = \frac{k \rho T}{\mu m_p} + \frac{1}{3} \alpha c T^4 \]  \hspace{1cm} (11.1)

where the first term is the gas pressure and the second the radiation pressure.

The energy losses were in Longair assumed to be described by an optically thick black-body. A more realistic model is obtained if we solve for the temperature using the diffusion equation with a realistic opacity. From Eq. (2.10) we have

\[ F = -\frac{4 \alpha c T^3}{3 \kappa \rho} \frac{dT}{dr} \]  \hspace{1cm} (11.2)

The optical depth is given by

\[ \tau = \int \kappa \rho dz \approx \kappa \rho H = \kappa \Sigma \]  \hspace{1cm} (11.3)

The flux should be equal to the vertically integrated viscous dissipation given by

\[ \int_0^H \frac{dE}{dt} dz = \frac{1}{2} \frac{3GM \dot{M}}{4\pi r^3} \left[ 1 - \left( \frac{R_s}{r} \right)^{1/2} \right] \]  \hspace{1cm} (11.4)

where the factor 1/2 accounts for the fact that only one half of the total flux emerges on each side of the disk. This should be equal to the flux given by Eq. (11.2), which we can approximate with

\[ F = -\frac{ac}{3\kappa \rho} \frac{dT}{dr} \approx \frac{ac}{3\kappa \rho} \frac{H}{3\tau} \]  \hspace{1cm} (11.5)

In this equation the temperature should be interpreted as the temperature in the center of the disk, \( z = 0 \). Combining these equations we get

\[ \frac{ac T^4}{3\tau} = \frac{3GM \dot{M}}{8\pi r^3} \left[ 1 - \left( \frac{R_s}{r} \right)^{1/2} \right] \]  \hspace{1cm} (11.6)

To illustrate the properties of an \( \alpha \)-disk we show the solution for the case of a optically thick, gas pressure dominated disk where the opacity is dominated by Kramer’s opacity. Kramer’s opacity, which is an approximation to the free-free and free-bound (i.e., photoelectric absorption) opacity, is given by

\[ \kappa = 5 \times 10^{24} \rho T^{-7/2} \text{ cm}^2\text{g}^{-1}. \]  \hspace{1cm} (11.7)
For this case the solution of the disk equations is given by

$$\Sigma = 5.2 \alpha^{-4/5} \dot{M}_{16}^{7/10} m^{1/4} r_{10}^{-3/4} f_1^{14/5} \text{ g cm}^{-2}$$

$$H = 1.7 \times 10^8 \alpha^{-1/10} \dot{M}_{16}^{3/20} m^{-3/8} r_{10}^{9/8} f_3^{3/5} \text{ cm}$$

$$\rho = 3.1 \times 10^{-8} \alpha^{-7/10} \dot{M}_{16}^{11/20} m^{5/8} r_{10}^{-15/8} f_1^{11/5} \text{ g cm}^{-3}$$

$$T_\epsilon = 1.4 \times 10^4 \alpha^{-1/5} \dot{M}_{16}^{3/10} m^{1/4} r_{10}^{-3/4} f_6^{6/5} \text{ K}$$

$$\tau = 1.9 \times 10^2 \alpha^{-4/5} \dot{M}_{16}^{1/5} f_4^{4/5}$$

$$\nu = 1.8 \times 10^{14} \alpha^{4/5} \dot{M}_{16}^{3/10} m^{-1/4} r_{10}^{-3/4} f_6^{6/5} \text{ cm}^2 \text{ s}^{-1}$$

$$v_r = 2.7 \times 10^4 \alpha^{4/5} \dot{M}_{16}^{3/10} m^{-1/4} r_{10}^{-1/4} f^{-14/5} \text{ cm s}^{-1}$$

\text{(11.8)}

where $f = \left(1 - (R_s/r)^{1/2}\right)^{1/4}$. The parameters are given by $m = M/ M_\odot$, $\dot{M}_{16} = \dot{M}/10^{16} \text{ g s}^{-1}$, and $r_{10} = r/10^4 \text{ cm}$.

From Eq. (11.8) we note several things:

1. The density, thickness of the disk and temperature are only weakly dependent on $\alpha$, while the optical depth and radial velocity are more sensitive to this parameter.

2. For typical values of $\dot{M}$ and $m$ the disk is indeed thin as we have assumed, with $H/r \propto \alpha^{-1/10} M^{3/20} r^{1/8}$

3. The radial velocity ($\sim 0.3 \text{ km s}^{-1}$ for $m = \dot{M}_{16} = r_{110} = 1$) is much less than the azimuthal ($v_\phi = 1150 m^{1/2} r_{10}^{-1/2} \text{ km s}^{-1}$). This justifies the Keplerian approximation for $v_\phi$.

Finally a word of caution: The scalings in Eq. (11.8) depend on the assumption that $\nu = \alpha c_s H$. The above scalings are therefore likely to change if we find a more realistic model for the viscosity.

The solution in Eq. (11.8) was based on the assumption that gas pressure dominates radiation pressure and that Kramer’s opacity is more important than electron scattering. The ratio of radiation pressure to gas pressure is

$$\frac{p_{\text{rad}}}{p_{\text{gas}}} = 2.8 \times 10^{-3} \alpha^{1/10} \dot{M}_{16}^{7/10} m^{-3/8} r_{10}^{-7/5} f_1^{7/5}$$

\text{(11.9)}

and one finds that Kramer’s opacity dominates electron scattering for

$$r \gtrsim 2.5 \times 10^7 \dot{M}_{16}^{2/3} m^{1/3} f_8^{8/3} \text{ cm}$$

\text{(11.10)}

In Fig. 21 we show the different regimes in the $r - \dot{M}$ plane, assuming that $\alpha = 1$ for the $p_{\text{rad}}/p_{\text{gas}} = 1$ boundary, and that $M = 1 M_\odot$. 47
Figure 21: Different cases for the thin disk solution for $M = 1 \, M_\odot$.

12 Gamma-ray Bursts

12.1 Historical overview

Gamma-ray bursts (GRB’s) were discovered in 1967 as a result of the Cold War. At that time the results were classified, and it was not until 1973 the first results were published. This immediately inspired a large number of more or less exotic theories, and in connection to the Texas conference on Relativistic Astrophysics in 1974 Ruderman could summarize more theories than discovered bursts at that time. Among these were more or less exotic candidates, like mini-black holes, white holes, comets falling down on neutron stars, etc. It is, however, worth noting that supernovae were already in 1974 proposed as a candidate by Colgate.

With the Compton Gamma-ray Observatory in 1991 it was found that the distribution on the sky was highly isotropic, indicating a cosmological origin, although an extended halo population could not be completely ruled
Figure 22: GRB distribution on the sky for bursts observed with BATSE

out (Fig. 22). CGRO also found that the bursts could roughly be divided into short ($\Delta t \lesssim 2$ s) and long ($\Delta t \gtrsim 2$ s) bursts, although the distribution of these formed a continuum.

A problem with CGRO was the fact that the localization could only be done within $\sim 3^\circ$. An optical identification, which requires a position within arc minutes, was therefore impossible.

A breakthrough came with the Dutch-Italian satellite Beppo-Sax which had both a gamma-ray trigger and an X-ray telescope which could localize the X-ray emission in connection to the burst within a few arc minutes. This allowed in 1997 the first optical identification of an afterglow from GRB970228. Shortly afterwards, GRB970508 was identified, and, moreover, the spectrum showed absorption lines from intervening galaxies up to $z = 0.835$, once and for all demonstrating that the GRBs are at cosmological distances.

In 1998 the GRB 980425 was found to coincide with the Type Ic SN 1998bw in ESO 184-G82 at $z=0.0085$. This SN was highly unusual from several points of view. The radio emission was the strongest seen among all
SNe (see Fig. ??). Also its optical luminosity was an order of magnitude higher than the typical Type Ic luminosity, indicating $M^{(56\text{Ni})} \sim 0.7 \, M_\odot$, which is larger than e.g., SN 1987A by a factor of ten. Finally the spectrum indicated an expansion velocity of $\gtrsim 60,000 \, \text{km s}^{-1}$, which was probably only a lower limit. Modeling of the radio observations showed that these could be well fitted with a synchrotron-self absorption spectrum of a source expanding with a Lorentz factor $\Gamma \sim 2$. This gave rise to the notion hypernovae. Although many in especially the SN community saw this as the confirming evidence for the SN – GRB connection, which was proposed already 1974 by Colgate and others, this GRB was extremely weak compared to typical GRBs.

More evidence, however, came from the afterglow light curves which in several cases showed a clear bump in the light curves, which was interpreted as a SN signature. Complete confirmation came with the identification of a SN 1998bw-like spectrum in the afterglow of the GRB 030329.

### 12.2 Summary of observations

#### 12.2.1 Prompt phase

In Fig. 23 we show a sample of burst profiles detected by BATSE. It is obvious that both the length of the burst and its light curve shape differ greatly. Some, like Triggers 1406 and 2571, have sharply rising bursts and then a smooth decay, while others like Trigger 1606 have highly irregular profiles.

The duration of the bursts vary from $10^{-2} \, \text{s}$ to $10^3 \, \text{s}$. As is seen from Fig. 23, many long bursts show considerable substructure, with peaks with a duration of the order of milliseconds or even less. The distribution of the durations show a clearly bimodal structure, with one peak at $\sim 0.2 \, \text{s}$ and one at $\sim 30 \, \text{s}$ (Fig. 24). Because of this, one usually divides the burst into short $\lesssim 2 \, \text{s}$ and long $\gtrsim 2 \, \text{s}$. The short account for $\sim 25\%$, but there may be selection effects which may increase the true fraction. This bimodal distribution has led some people to the suggestion that this represents two different physical mechanisms for the bursts. We are coming back to this later.

The spectra of the prompt emission can be described as two power laws with a break between, $dN(E)/dE \propto E^{-\alpha}$ where $\alpha$ is the photon number index. At energies less than the peak energy, $E_p$, $\alpha \approx 1 \pm 1$, while above $E_p$ the bursts have a wide range of $\alpha \approx 1 - 4$. The fact that the $EF(E)$ spectrum shows a peak in the gamma-ray range, implies that for most GRBs most of
Figure 23: Examples of light curves observed with BATSE.
the energy of the burst is really coming out as gamma-rays. Recently, many bursts have, however, been discovered which have their peak energy in the X-rays.

The range in $E_p$ is very large ranging from MeVs down to tens of keVs. Unfortunately, both at high and low energies selection effects makes this highly uncertain.

There is a correlation between the peak energy and the duration of the burst, so that short bursts in general have harder spectra than long bursts. This can be seen in the hardness ratio defined as the ratio of the fluency (time integrated flux) in the 100-300 keV channel divided by that in the 50 – 100 keV channel of the BATSE instrument (Fig. 25). Clearly, the short bursts have considerably harder spectra. The transition between the two groups is, however, smooth.

12.2.2 Afterglow phase

GRB 970508 was the first GRB to show an afterglow emission in the radio as well as in the optical. This allowed first of all an identification of the object at these wavelength, and secondly a very valuable diagnostic of the global properties of the GRB. This includes such parameters as the total energy, the Lorentz factor of the expanding matter, magnetic fields and particle energies, as well as information about the environment of the GRB. In particular, this has allowed a direct identification of the type of object
Figure 25: Hardness – duration correlation of BATSE bursts. The HR is defined as the ratio of the fluency in the 100-300 keV channel divided by that in the 50 – 100 keV channel (from Qin et al. 1999)
which is responsible for the GRB. As examples of afterglow light curves in
the X-ray range we show in Fig. 26 a sample of recently observed GRBs
with SWIFT.

The afterglow phase is a direct relativistic version of the Sedov solution
for a supernova remnant. In the same way as for these the dynamics reflects
the properties of the radiative emission and spectrum. Before discussing this,
we, however, consider a few very basic constraints which can be derived from
observations of the prompt phase.

12.3 The necessity of relativistic expansion

First assume that the source is non-relativistic. The fluency is then given by
\[ F = \frac{L \Delta t}{4\pi D^2} \], where \( \Delta t \) is the time scale of the burst, \( L \) the luminosity
and \( D \) the luminosity distance to the GRB. If we assume that the source has
a radius $R$, the energy density is $L/4\pi R^2 c$. As an estimate we take $R \sim \Delta t$. Further, we assume that a fraction $f_p$ of the photons have energies above the pair creation threshold, $\sim 2m_e c^2$, the density of energetic photons is $n_\gamma = f_p L/4\pi R^2 m_e c^3$. The optical depth to pair production is therefore

$$\tau_{\gamma\gamma} = \frac{\sigma_T f_p L}{4\pi R m_e c^3} = \frac{\sigma_T f_p F D^2}{R \Delta t m_e c^3} = \frac{\sigma_T f_p F D^2}{(\Delta t)c^2 m_e c^3} \quad (12.1)$$

As a typical value for the fluency we take $F \sim 10^{-6} \text{erg cm}^{-2}$ and $D \sim 2000$ Mpc, corresponding to a total energy of $5 \times 10^{50}$ ergs. If we take $\Delta t \sim 0.01 \text{s}$ we get

$$\tau_{\gamma\gamma} = 3 \times 10^{14} f_p \frac{F}{10^{-6} \text{erg cm}^{-2}} \left(\frac{\Delta t}{0.01 \text{s}}\right)^{-2} \quad (12.2)$$

Therefore for any reasonable values of $f_p$ the source would be extremely optically thick to pair production and would therefore show a thermal spectrum, contrary to the observations.

This paradox is solved if the source is expanding relativistically with a large Lorentz factor, $\Gamma$. This has several consequences which help in the right direction.

First, if the source is moving towards us with a velocity $v$, the observed time interval, $dt_{\text{obs}}$ between two photons emitted in an interval $dt_{\text{em}}$ will be smaller by a factor $2\Gamma^2$. To see this we consider a photon emitted from the shell at a radius $r_1$ from the origin at a time $t_1 \text{em}$ in the GRB frame, and at an angle $\theta$. The time when it will arrive to the observer is therefore $t_1 \text{obs} = t_1 \text{em} + (D - r_1 \cos \theta)/c$. Now, let a second photon be emitted at a time $t_2 \text{em} + dt_{\text{em}}$. The radius will now be $r_2 = r_1 + v dt_{\text{em}}$, and the time when it will be observed is therefore $t_2 \text{obs} = t_2 \text{em} + (D - r_2 \cos \theta)/c = [D - (r_1 + v dt_{\text{em}}) \cos \theta]/c$. The time interval it will be received in is therefore

$$dt_{\text{obs}} = dt_{\text{em}} - \beta dt_{\text{em}} \cos \theta = dt_{\text{em}} (1 - \beta \cos \theta) \quad (12.3)$$

where $\beta = v/c$.

Because $v \approx c$ it is more useful to write this in terms of the Lorentz factor. For this we note that $\Gamma^2 = 1/(1 - \beta^2) = 1/[(1 + \beta)(1 - \beta)] \approx 1/[2(1 - \beta)]$. If we now assume that $\cos \theta \approx 1$, we can write Eq. (12.3) as

$$dt_{\text{obs}} = \frac{dt_{\text{em}}}{2\Gamma^2} \quad (12.4)$$

If the source is expanding with constant velocity the true size is therefore not $dt_{\text{obs}}$ but $2\Gamma^2 ct_{\text{obs}}$. 55
Note that Eq. (12.4) is not the result of a Lorentz transformation, but is only a result of the fast expansion and the finite velocity of light.

Secondly, for a relativistically expanding source the radiation we receive will be blue shifted by a the Doppler effect which gives a factor $\Gamma$ higher frequency. Therefore, the number of photons above the pair production threshold will decrease by a factor $\Gamma^2 \alpha$.

Putting everything together, one gains a factor of $\Gamma^2 (1 + \alpha)$ from the relativistic motion. With $\alpha \sim 2$ this becomes $\sim \Gamma^6$. The Lorentz factors needed to have $\tau_{\gamma\gamma} \ll 1$ are therefore in the range $\Gamma \sim 100 - 1000$.

Further evidence of relativistic expansion comes from radio observations of interstellar scintillations in GRB light curves. An example of this is shown in Fig. 27 for GRB 970508. During the first $\sim 50$ days the radio flux showed large excursions, which later decreased, consistent with that expected for an expanding source. From the size of the plasma fluctuations the angular extent of the radio emission could be estimated, and one found that the source must have had a size of $\gtrsim 10^{17}$ cm, showing that the expansion was close to the velocity of light.

Figure 27: Radio light curve for GRB970508 at 8.46 GHz. Note the rapid fluctuations due to interstellar scintillations in the light curve during the first $\sim 50$ days (Frail 2003).
The fact that we need a highly relativistic expansion means that the mass involved in this must be very small, since

\[ E \sim \Gamma M c^2 \] (12.5)

which means that

\[ M \approx 5 \times 10^{-6} \left( \frac{\Gamma}{10^5} \right)^{-1} \left( \frac{E}{10^{52} \text{ergs}} \right) \, M_\odot \] (12.6)

This means that the explosion has to have a very small fraction of baryons to photons.

12.4 General scenario for the prompt and afterglow emission.

Most models for the prompt emission, as well as the afterglow, do not specify the way the explosion takes place. The only assumption is that a very large amount of energy is released in either a spherical explosion or, as we will discuss below, in a narrow conical jet. This is both the strength and weakness of the model. On the one hand it is free of assumptions about this early not well understood phase. On the other hand, the model gives very little information about this crucial stage.

In it simplest version one releases a large amount of energy in a medium and let this expand. Depending on the structure of the medium, like radial density variation, asphericity and rotation, etc, the explosion may either be spherical or, more realistically, confined to a jet. In the currently popular models the latter is usually assumed.

The prompt emission may be produced in two quite different alternatives (see Fig. 29). In one type of models it is a result of the interaction of the blast wave with the external medium, in a similar way to what happens in a supernova remnant. The properties of the emission is therefore sensitive to the details of the circumstellar medium, such as clumping and the presence of a stellar wind from the progenitor. This model is usually called the external shock model.

In the other type of models the prompt emission is caused by the fact that it is likely that the ‘central engine’ does not have a steady energy and momentum output as function of time. Instead, the Lorentz factor may e.g., vary with time depending on the details of the energy production mechanism. This variation will lead to a situation where you may have matter with a higher Lorentz factor ejected after that of lower. The more energetic
ejection will therefore at some point catch up with the less energetic, and a shock wave will form in the outflow itself. The energy release is in this type of models therefore internal to the outflow and the models are therefore referred to as the internal shock model.

An important clue to the the cause of the prompt phase is the rapid variations in intensity on a time scale of milli-seconds seen in most bursts. There are basically two possibilities to create this.

In the external shock model the variations are caused by encounters of the relativistic blast wave by a large number of clumps in the circumstellar medium. A problem for this model is, however, that the shock emission resulting from this does not react fast enough to the clumping but instead smooths the variation with time. The internal shock model has no problems in this respect, since the variations in the radiative flux will directly reflect the variation in the outflow. For this reason it is the currently favored model for the prompt emission. After the central engine has been switched off and these internal variations have had time to set up a smooth outflow, one expects the internal energy release to switch off. At that point the

Figure 28: Schematic representations of the different stages in the evolution of a GRB.
interaction with the external medium takes over, and the GRB has entered the afterglow phase. The details of the model, in particular the reason for the variable Lorentz factor, are, however not at all clear.

The emission in the afterglow phase is better understood. In principle it is just a relativistic version of the blast wave for a supernova remnant. In the same way as this the kinetic energy of the outflow is converted into thermal energy behind the shock. One can then formulate shock conditions similar to the non-relativistic, which gives the relation between the conditions in front and behind the shock (for details see the Appendix). For an adiabatic expansion of the blast wave one can then find a similarity solution for the dynamics, similar to the Sedov solution, which describes the expansion of the blast wave in the surrounding medium. Compared to the Sedov solution this is complicated first by the relativistic effects from the Lorentz transformations and secondly by the difference in the observed time evolution and the time evolution in the frame of the GRB, as we discussed in the previous
section. Without any details we the result is

\[ R \approx \left( \frac{9E}{2\pi n_1 m_e c} \right)^{1/8} t_{\text{obs}}^{1/4} \]  

(12.7)

or

\[ R \approx 4 \times 10^{17} \left( \frac{E}{10^{52} \text{ ergs}} \right)^{1/8} \left( \frac{n_1}{1 \text{ cm}^{-3}} \right)^{-1/8} \left( \frac{t_{\text{obs}}}{\text{days}} \right)^{1/4} \text{ cm}. \]  

(12.8)

The Lorentz factor of the shocked gas behind the blast wave, with parameters considered to be typical for a GRB, is given by

\[ \Gamma_2 \approx 4.4 \left( \frac{E}{10^{52} \text{ ergs}} \right)^{1/8} \left( \frac{n_1}{1 \text{ cm}^{-3}} \right)^{-1/8} \left( \frac{t_{\text{obs}}}{\text{days}} \right)^{-3/8} \]  

(12.9)

The energy, as well as the density, can vary by large factors, and can in particular be considerably higher respectively lower. It is therefore conceivable that larger Lorentz factors than the indicated are at hand in some cases. Note, however, especially the fairly strong dependence on the observer time. Lorentz factors of 100–1000 are therefore likely to be present during the first minutes and hours after the burst, as is needed from the discussion earlier.

The derivation of these relations are given in the Appendix for those interested.

12.5 Afterglow spectra

For synchrotron radiation an electron in a magnetic field, \( B \), with energy \( \gamma m_e c^2 \) radiates a total power \( P = 4/3\pi T B^2 \gamma^2 /8\pi \), at a frequency

\[ \nu_0 = \frac{eB\gamma^2}{2\pi m_e c} = \nu_B \gamma^2 B, \]  

(12.10)

where \( \nu_B = 4.2 \times 10^6 \) Hz. The spectral distribution can be approximated by

\[ P(x) \approx x^{1/3} \exp(-x) \]  

where \( x = \nu/\nu_0 \) (e.g., Rybicki & Lightman). This is the frequency in the comoving frame. The frequency in the observer frame is \( \nu_{\text{obs}} = \Gamma_2 \nu_0 \). Both observationally and theoretically there are strong reasons for assuming that the non-thermal electron spectrum is given by a power law, \( dn(\gamma)/d\gamma \propto \gamma^{-p} \), where \( p \sim 2 \).

The spectrum of the afterglow is characterized by a number of power law segments, separated by several breaks. The frequencies of these breaks correspond to the minimum energy of the electron distribution, \( \nu_{\text{min}} = \nu_B \gamma_{\text{min}}^2 B \),
the energy where synchrotron cooling becomes important, $\nu_c = \nu_B \gamma_c^2$, and the frequency where synchrotron self-absorption becomes important $\nu_{SSA}$.

Let us for a moment ignore the synchrotron self-absorption, and also assume that $\nu_{\min} < \nu_c$. Below $\nu_{\min}$ the radiation will be dominated by electrons close to $\gamma_{\min}$. The spectrum from these will be the same as a mono-energetic spectrum with $\nu < \nu_0 = \nu_{\min}$. i.e., $F_\nu \propto P(\nu/\nu_{\min}) \propto (\nu/\nu_{\min})^{1/3}$. Between $\nu_{\min}$ and $\nu_c$ the spectrum will be $F_\nu \propto (\nu/\nu_{\min})^{-(p-1)/2}$, the usual synchrotron spectrum. Finally, for $\nu > \nu_c$ cooling is important, and $F_\nu \propto (\nu/\nu_c)^{-p/2}$.

Let us now include synchrotron self-absorption. We then have to consider two cases. Let us first assume that $\nu_{\min} < \nu_{SSA}$. For $\nu_{\min} < \nu < \nu_{SSA}$ the spectrum will then be $F_\nu \propto (\nu/\nu_c)^{5/2}$, characteristic of an optically thick source. For $\nu < \nu_{\min}$ the spectrum will, however, be somewhat flatter $F_\nu \propto (\nu/\nu_{\min})^2$. The reason for this can be understood if we write the spectrum in the optically thick Rayleigh-Wien limit as $F_\nu = 2\nu^2 E_{\text{mean}}/c^2$, where $E_{\text{mean}}$ is the mean energy of the radiating particles. For a thermal distribution $E_{\text{mean}} \sim kT$, while for a non-thermal $E_{\text{mean}} \propto \gamma_{\text{mean}} \propto (\nu_{\text{mean}}/B)^{1/2}$. In this case $F_\nu \propto \nu^2 \gamma_{\text{mean}} \propto \nu^{5/2}/B^{1/2}$. However, if $\nu < \nu_{\min}$, then the electrons at $\gamma_{\min}$ are doing most of the absorption and emission, so that $\gamma_{\text{mean}} = \gamma_{\min}$ and $F_\nu \propto \nu^2 \gamma_{\min} \propto (\nu_{\min}/B)^{1/2} \nu^2$. In Fig. 30 we summarize the different sections of the spectrum.

To demonstrate this with real observations, we show in Fig. 31 a fit to the broad band spectrum of GRB970508, which is one of the best examples of this kind of fit. From the fit, in combination with the time evolution,
values of $n_1, E, \varepsilon_e, \varepsilon_B, \gamma_{\min}, p$ can be derived. The value of $p$ is obtained directly from the spectral slope and typical values are $p \approx 2.1$, which is in good accordance with theoretical expectations. The other parameters are more uncertain and model dependent.

From the fit, one can determine the frequencies of the spectral breaks $\nu_{SSA}$, $\nu_{\min}$ and $\nu_c$, as well as the flux at the peak, $F_{\nu}(\nu_{\min})$. The value of $p$ is obtained directly from the spectral slope, and typical values are $p \approx 2.2$, which is in good accordance with theoretical expectations. The remaining parameters characterizing the blast wave $n_1, E, \varepsilon_e, \varepsilon_B$ can then be derived. Assuming a constant external density Panaitescu & Kumar get $n_1 = 0.1 - 30 \text{ cm}^{-3}$, $E \sim (1 - 5) \times 10^{50} \text{ ergs}$, $\varepsilon_e \approx 0.1$, $\varepsilon_B \approx 10^{-4} - 0.1$. The energy is corrected for beaming, as will be discussed below. These numbers should be taken with caution, since they depend on uncertain observations, as well as questionable assumptions.

Note that we have until now not made any assumptions about the time evolution of the remnant. This, however, enters in the time evolution of the frequencies of the spectral breaks, and therefore depend on the density profile of the environment and whether the blast wave is adiabatic or not.

### 12.6 Jet steepening

The fact that the energy, assuming isotropic emission, is so enormous has lead to the suggestion that the relativistic outflow occurs in two narrow
jets, in analogy with e.g., jets from compact radio galaxies. This is also motivated from hydrodynamical models for the GRB, as will be discussed later. Because of relativistic abberation, the radiation emitted at an angle $\theta'$ relative to its velocity in the rest frame, will be seen to be emitted at an angle $\theta$ given by

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}$$

(12.11)

(e.g., Rybicki & Lightman §4.1.3, or the similar effect for synchrotron emission in the electron and observer frames). Consider a light ray emitted at $\theta' = 90^\circ$. Then $\cos \theta = \beta$. Using $1 - \beta \approx 1/2\Gamma^2$, and $\cos \theta \approx 1 - \theta^2/2$, we find $\theta \approx 1/\Gamma$ for $\Gamma \gg 1$.

Therefore, for relativistic velocities the radiation is seen only within an angle $\theta \sim 1/\Gamma$. As long as the jet opening angle is larger than this, there is no difference between a spherical shell and a jet. However, as the jet is slowing down there will be a point when this condition is no longer true. Because only part of the emitting cone will now be filled this leads to a steepening of the light curve of the afterglow. Therefore, if one can determine when this occurs one can from the afterglow model estimate the value of $\theta \sim 1/\Gamma$ (see Eq. (C.11)), if one from the spectral modeling has determined the other parameters. Knowing $\theta$, one can now determine the correct total energy as $E = 2\theta^2 E_{iso}/4\pi$ (for a two-sided jet), where $E_{iso}$ is the energy assuming an isotropic shell.

In Fig. 32 we show the optical light curves of GRB 990510. At $\sim 1$ day there is a clear steepening in the light curves of all colors. This has now been done for a number of afterglows, and in Fig. 33 we show a distribution of isotropic and corrected total energies. Typical jet angles are $\sim 10^\circ$. It is here seen that while the isotropic energies are in the range $10^{52} - 10^{54}$ ergs, the beam-corrected energies are $\sim 5 \times 10^{50}$ ergs, with a small dispersion. This has lead to the suggestion that one can use GRBs as standard candles in the same way as TypeIa supernovae. Because of many systematic effects this is in my view optimistic.

### 13 GRB Progenitors

Up to now we have made no assumptions about the nature of the exploding object, but only assumed an instantaneous injection of a large amount of energy with a large $E/M_0c^2$. To explain the large energies involved $\sim 10^{51} - 10^{52}$ ergs it is, however, obvious that the formation of some kind of compact object is involved. This can either be a neutron star or a black hole. There
Figure 32: Optical light curves of GRB 990510 in the V, R and I bands (Harrison et al. 1999).

Figure 33: Distribution of isotropic end corrected energies (Frail et al. 2001).
are then two main classes of scenarios, which have quite different progenitors. The physics involved in the generation of the energy may, however, be fairly similar. We will now discuss these one by one.

13.1 The supernova - GRB connection

Supernovae have from the theoretical point of view for a long time been proposed as an origin for GRBs. When the first afterglows were identified, it was also noted that these in most cases were in the central regions of star forming galaxies, typical of massive stars. Direct evidence for a connection between these was, however, lacking. This changed when in April 1998 the error box of GRB 980425 was found to coincide with the supernova SN 1998bw in the galaxy ESO 184-G82 with a very low redshift, 2550 km s\(^{-1}\) or \(z=0.0085\). The supernova which was a Type Ic SN, was very remarkable from several points of view. The radio emission from the supernova was found to be more luminous than any other radio SN, and was well fitted by a synchrotron self-absorption spectrum. From modeling of the radio emission the expansion velocity of the emitting material was found to have a Lorentz factor of \(\Gamma \sim 2\). Also the optical spectrum showed very broad, smooth features indicating an expansion velocity of at least 60,000 km s\(^{-1}\).

The luminosity of the SN was nearly a factor of ten larger than a typical Type Ic SN, and close to that of Type Ia’s. The light curve indicated a total \(^{56}\)Ni mass of \(\sim 0.5 \, M_\odot\), much higher than that in e.g., SN 1987A. The gamma-ray luminosity was, however, about four orders of magnitude less than a typical GRB, \(\sim 5 \times 10^{47}\) ergs. This has lead to some doubt about the GRB-SN connection in this case. The coincidence of the SN and GRB as well as the remarkable properties of the SN, makes it in my view, however, completely clear that the GRB and the SN really originated from the same object.

In addition to this direct evidence there has for a number of GRBs been seen evidence for a bump in the light curve of the afterglow. While the early evolution in most cases follows a power law, there has been several examples where a red bump has been seen in the light curve at \(\sim 20\) days (Fig. 34). The luminosity of these bumps as well as the shape and color are roughly consistent with that of SN 1998bw, indicating that it really is the emission from the SN which is seen.

Besides SN 1998bw, the most direct evidence for the SN/GRB connection came from GRB 030329. This was by GRB standards an extremely nearby GRB with \(z = 0.168\). As was immediately recognized by several groups, this was a unique opportunity of getting high S/N spectra of the
Figure 34: Light curves of the afterglow of GRB011121 obtained with HST (triangles) and ground based telescopes (diamonds). Note the bump in the light curve at 10-30 days, consistent with that from a of SN 1998bw, dimmed by $\sim 55\%$ (Bloom et al 2002).
Figure 35: Spectral sequence of GRB030329/SN2003dh with VLT (Hjorth et al 2003). Note the power law spectrum on April 3 and the gradually stronger supernova contribution. The dashed line shows the spectrum of SN 1998bw at an age of 33 days.
afterglow during the first months. While the first spectra showed basically a power law spectrum with $F_\nu \propto \nu^{-1.2}$, there was after $\sim 8$ days a clear excess emission above a power law fit (Fig. 35). This component became increasingly stronger, and when the power law spectrum seen during the first days was subtracted it was found that this coincided almost perfectly with that of SN 1998bw. The supernova consequently got the designation SN 2003dh. In addition to this there has been several other GRBs where there is strong evidence for an underplaying supernova. With GRB980425, GRB030329 and these other cases, the SN/GRB connection is now firmly established.

Note, however, that the optically identified GRBs all belong to the long GRBs. Because there is some evidence from the distribution of the durations that there may be two different populations of progenitors, it is fair to say that the SN–GRB connection is only established for the long bursts. The short could have a different class of progenitors.

13.2 The GRB environment

13.3 Collapsars

The collapsar model for GRBs is based on the partial failure to produce an explosion from models of core collapse SNe. However, to get a gamma-ray burst several special properties of the collapsing star are likely to be needed. This is also indicated by the fact that only a very small fraction of all SNe produce GRBs. From the beaming angle, corresponding to a solid angle of $\Omega \sim 0.03$, we observe only every $\leq 200$ of all GRBs. More detailed estimates of this factor vary between $75 - 500$. The total GRB rate (including the ones with beaming away from us) is estimated to be $\sim 33$ Gpc$^{-3}$ year$^{-1}$. The typical rate should then be one GRB per $\sim 3 \times 10^5$ years for a typical galaxy. Therefore, only a fraction of one per $\sim 3 \times 10^3$ SNe will become a GRB.

In the standard GRB scenario the main ingredients is a rapidly rotating stellar core, and a low mass or absent stellar envelope. The former is needed to produce a jet, while the latter is needed to get the jet out of the star.

The main parameters of the collapsing core are the specific angular momentum, $j = J/M$. During the first seconds a centrifugally supported disk forms with interior to $R \approx j^2/GM$. For reasonable values of $j$ this is $\sim 100 - 200$ km. Because the centrifugal support is much smaller in the polar direction the matter in this direction continues to accrete onto the black hole, until this region is nearly empty (Fig. 36). The density contrast between the equatorial disk, where the density is $\sim 10^9$ g cm$^{-3}$, and the
Figure 36: Density in the center $\sim 7$ s after collapse. Inside of $\sim 200$ km the centrifugally supported torus can be seen. In the polar direction the density is very low because the lack of centrifugal support has emptied this region. (MacFadyen & Woosley 1999)
polar direction will therefore be very large.

Most of the energy losses from the disk will, because of the high temperature, \( \sim 10^{10} \) K, be in the form of neutrinos. Because the disk dominates the neutrino luminosity, the neutrino radiation field will be highly anisotropic. Neutrino annihilation, \( k, \nu + \bar{\nu} \rightarrow e^- + e^+ \), above the disk may then produce electron-positron pairs in this region. These can then give rise to a jet perpendicular to the disk. The details of this mechanism are, however, uncertain.

Another suggestion uses some kind of electromagnetic extraction of the energy from the disk in a similar manner as a pulsar. A magnetic field anchored in the disk and treading the black hole horizon can tap the black hole on rotational energy by the so-called Blandford-Znajek mechanism. The total amount of rotational energy is in principle enormous, \( \sim 10^{54} \) ergs, but again, this mechanism is not worked out in sufficient detail for a proper evaluation of its merits.

In some way or another a large amount of energy is likely to be deposited in the polar directions above the disk on a time scale of the order of 1-100 s. This is the starting point of the two jets along the rotational axis of the star.

As the jet is launched from the center, it propagates outwards through the star. Fig. 37 shows a simulation of this from the inner region up to the point when it interacts with the circumstellar medium in the form of a wind from the progenitor. The radius of the Wolf-Rayet progenitor is in this model \( 8 \times 10^{10} \) cm and the He-core mass 15 M\(_\odot\).

While inside the star, the narrow jet will be preceded by a cocoon, consisting of shocked material from the stellar core and envelope, as well as the shocked jet. This cocoon propagates through the star with a sub-relativistic velocity, \( \sim (5 - 10) \times 10^4 \) km s\(^{-1}\), although the jet itself is relativistic with \( \Gamma \sim 10 \). As it penetrates through the surface of the star the cocoon spreads in angle and also accelerates down the steep density at the surface. This results in a Lorentz factor of \( \sim 5 - 10 \) for the cocoon and an angular extent of \( \sim 30^\circ \).

In Fig. 38 we show the density and Lorentz factor at the final epoch of the model above. Although only \( \sim 20 \) at the time of the jet break-out, the final Lorentz factor in the jet reaches \( \gtrsim 100 \) as the internal energy is converted to kinetic energy by the adiabatic expansion. A most important thing to note in the figure is the highly variable Lorentz factor in the jet. As the faster material will catch up with the slower, internal shocks in the jet will form, explaining the initial burst. This can explain the prompt burst as we discussed earlier.
Figure 37: Simulation of jet propagation and break-out. The different panels show the Lorentz factor and density at six epochs, 5, 10, 12, 20, 40, and 70 s. (from Zhang, Woosley, Heger 2003)
The cocoon mentioned above is interesting because although it only contains a minor fraction of the total energy, it has a factor of 5–10 larger angular extent than the jet itself. The solid angle, and thus the probability of observing it, is therefore a factor of 25–100 larger. The lower Lorentz factor means that the radiation from the cocoon should be considerably softer. It has been proposed that this may explain the so called X-ray flashes (XRFs), which has most of their energy in the X-ray rather than gamma-ray domain.

The collapsar model is a very likely candidate to explain the long bursts. The duration of the burst is set roughly by the time scale of the launch of the jet. It is, however, difficult to see that this can be much shorter than seconds, and the model has therefore problems explaining the short bursts. In principle, it is, however, possible also to get very short bursts from the interaction of the jet and the head of the cocoon.

13.4 Neutron star mergers

We know that binary neutron stars exist, as the famous case of the Hulse-Taylor pulsar PSR1913+16 shows. This system will decay by gravitational radiation on a time scale of \( \sim 10^8 \) years. Because both stars have a mass close to 1.4 \( M_\odot \), the result will most likely be a black hole, unless a very large fraction of the mass is expelled. The energy release in connection to
this may be very large, comparable to that in an ordinary core collapse supernova. The time scale will be of the order of milliseconds. This has led to the suggestion that merging neutron stars may have something to do with GRBs, and was for a long time the most popular GRB model. As we have seen, there is now compelling evidence that the long GRBs are connected to supernovae. This evidence does, however, not apply to the short GRBs. In particular, this models has some properties which can easier explain the short time scales connected with this class of GRBs.

As the neutron stars spiral in they will lose more and more of the orbital energy by gravitational radiation. The final merger will occur on a time scale of the order of milliseconds. Because of the large angular momentum the tidal forces will distort and tear apart the stars, and a flattened, disk like configuration will form (Fig. 39). While most of the mass results in a black hole of mass $\sim 2.5 \, M_\odot$, a substantial fraction, $\sim 0.1 - 0.2 \, M_\odot$, will stay in the form of an extremely hot accretion disk. The temperature of this will be $\sim 10^{10}$ K, and it will therefore loose most of its internal energy as neutrinos. The accretion rate will be $\sim 1 \, M_\odot \, s^{-1}$, so the disk will have a life time of $\sim 0.1 - 0.2 \, s$. The total energy in the neutrinos will be of the same order as for a core collapse SN, $\sim 10^{53}$ ergs, more than enough to feed a GRB with a moderate amount of beaming. The fundamental problem is just how to convert the neutrino energy into photons.

Similar to the collapsars, there are two main mechanisms which have been proposed for this, neutrino annihilation or electromagnetic extraction. The neutrinos produce electron-positron pairs, which give rise to a jet perpendicular to the disk. This then convert its kinetic energy into heat by internal shocks, giving rise to a gamma-ray burst. Because the neutron star binary is not expected to have any circumstellar medium, the afterglow is expected to be the result of interaction of the outflow with the interstellar gas, having a constant density.

The main problem with this mechanism is that detailed simulations show that the efficiency of the neutrino pair annihilation is relatively inefficient. The energy converted into pairs is $\sim 5 \times 10^{49}$ ergs, which may be too low. This is especially true since it is difficult to obtain the narrow beaming suggested by the afterglow observations. Although highly uncertain, the MHD extraction of energy may be the most promising, but also most complex to calculate.

Summarizing this model, it has the virtue of being based on events which we know will take place, and that the total energy available is sufficient. The drawbacks is the difficulty of converting this energy to photons. In addition, the frequency of these mergers is highly uncertain, although estimates give
Figure 39: Evolution of the neutron star binary at different epochs after the start of the simulation. The contours show the density and temperature, while the arrows show the velocity field. (Ruffert & Janka 2001).
a rate of one merger per $\sim 10^6$ years for a typical $L_*$ galaxy.

Recently, there has been observations of some short GRBs with SWIFT which have given support to this progenitor scenario. GRB 050509B, GRB 050709 and GRB 050724 were the first short GRBs to have an X-ray afterglow and therefore allow a precise localization of the GRB. For two of these optical afterglows were also found. The redshift of the galaxies were all comparatively low, 0.16 - 0.25. In two of the cases the host galaxies were ellipticals, while in the other it was a star forming dwarf galaxy. Compared to the long bursts, the gamma-ray luminosity is down by 2-3 orders of magnitude.

No indication of a supernova was found in the optical afterglows and the modeling of the spectrum and light curves of the afterglows indicate very low density environments. Both these results, as well as the location in two of the cases in elliptical galaxies, are consistent with what would be expected for a merger of two compact objects. While the neutron –neutron star merger is the most favored, also a black hole – neutron star merger is a possibility.

It is likely that SWIFT will discover several more of the short bursts, and that we will therefore get considerably more information about this class of GRBs in the near future.
A Equations of relativistic hydrodynamics

This Appendix is NOT included in the course. Just for the really interested!! See Weinberg Chap. 2.10 and Landau & Lifshitz, Fluid Dynamics 1985 for more details.

When we discuss the dynamics and radiation from the GRB there are three reference frames which are of interest. The rest frame of the exploding star, the comoving frame of the expanding gas and the reference frame of the observer.

An example is the time interval of a process as measured in the comoving frame, $dt_c$, and that of the GRB, which are related as

$$dt_{\text{GRB}} = \gamma dt_c \tag{A.1}$$

In general when we want to relate measurements in different reference systems it is convenient to see these as transformations between different four-vectors. An example of such a four-vector is $x^\mu = (x, y, z, t)$. If the coordinates of an event in a system, K, moving with velocity $v$ relative to another system K’ along the x-axis is $x^\mu = (x, y, z, t)$, then the coordinates in the system K’ are given by

$$x'^\mu = \sum_{\nu=1}^{4} \Lambda^\mu_\nu x^\nu \tag{A.2}$$

where $\nu = 1, 2, 3$ denote the space components and $\nu = 4$ the time component. The matrix $\Lambda^\mu_\nu$ is called the Lorentz boost, and is for the case of motion only along the x-axis given by

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & \gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta & 0 & 0 & \gamma \end{pmatrix} \tag{A.3}$$

Here $\beta = v/c$ and $\gamma = 1/\sqrt{1-(v/c)^2}$.

The energy density and pressure transform as components of the of the energy–momentum tensor, $T^\mu\nu$, which in the rest frame is given by

$$T^\mu\nu = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{A.4}$$
To transform this into a system moving along the \(x\)-axis, we apply two Lorentz boosts,
\[
T^{\mu\lambda} = \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} \Lambda_{\mu}^{\kappa} \Lambda_{\nu}^{\lambda} T^{\mu\nu}.
\] (A.5)

We therefore obtain
\[
T^{\mu\nu} = \begin{pmatrix}
\gamma^2 (\epsilon \beta^2 + p) & 0 & 0 & \gamma^2 \beta (\epsilon + p) \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
\gamma^2 \beta (\epsilon + p) & 0 & 0 & \gamma^2 (\epsilon + \beta^2 p)
\end{pmatrix}
\] (A.6)

**Exercise:**
Show Eq. (A.6)!

The hydrodynamic equations are given by the divergence of the energy momentum tensor,
\[
\sum_{\nu=1}^{4} \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0
\] (A.7)

In spherical symmetry we get
\[
\frac{\partial}{\partial t} \gamma n + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \gamma \beta n = 0
\] (A.8)
\[
\frac{\partial}{\partial t} \gamma^2 \beta (\epsilon + p) + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \gamma^2 \beta^2 (\epsilon + p) + \frac{\partial p}{\partial r} = 0
\] (A.9)

In addition we have the conservation of the particle number. The particle number in the rest frame is \(n\). This is the time component of the particle current four-vector. To transform this into an arbitrary frame we apply again a Lorentz boost \(J^{\mu} = \sum_{\nu=1}^{4} \Lambda_{\nu}^{\mu} J^{\nu}\), or
\[
J^{\mu} = (\gamma \beta n, 0, 0, \gamma n)
\] (A.10)

Taking the four-divergence in spherical geometry we get
\[
\frac{\partial}{\partial t} \gamma n + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \gamma \beta n = 0
\] (A.11)
B Relativistic shocks

Denote the postshock medium by 2 and the preshock medium by 1. As in the non-relativistic case, the relativistic shock conditions are obtained by integrating Eqns. (A.8), (A.9) and (A.11) over an infinitesimal radial distance across the shock. All the time derivative terms then go to zero, while the radial derivatives result in the jump conditions

\[ \gamma_1 \beta_1 n_1 = \gamma_2 \beta_2 n_2 \] (B.1)

\[ \gamma_1^2 \beta_1 (\epsilon_1 + p_1) = \gamma_2^2 \beta_2 (\epsilon_2 + p_2) \] (B.2)

\[ \gamma_1^2 \beta_1^2 (\epsilon_1 + p_1) + p_1 = \gamma_2^2 \beta_2^2 (\epsilon_2 + p_2) + p_2 \] (B.3)

The energy densities and pressure refer to the proper values before and after the shock, respectively. All velocities are relative to the reference frame of the shock.

From the second and third equation one can solve for \( \beta_1 \) and \( \beta_2 \). After some algebra, most easily done by setting \( \beta_1 = \tanh \phi_1 \) etc., one obtains

\[ \beta_1 = \left[ \frac{(p_2 - p_1)(\epsilon_2 + p_1)}{(\epsilon_2 - \epsilon_1)(\epsilon_1 + p_2)} \right]^{1/2} \] (B.4)

\[ \beta_2 = \left[ \frac{(p_2 - p_1)(\epsilon_1 + p_2)}{\epsilon_2 - \epsilon_1)(\epsilon_2 + p_1)} \right]^{1/2} \] (B.5)

The relative velocity of the preshock and postshock gas is given by the law of addition of velocities \( \beta = (\beta_1 - \beta_2)/(1 - \beta_1 \beta_2) \), so

\[ \beta = \left[ \frac{(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + p_2)(\epsilon_2 + p_1)} \right]^{1/2} \] (B.6)

Because the medium into which the shock is propagating is at rest, the Lorentz factor of the shock as seen by an observer is \( \Gamma_s = \gamma_1 \), and \( \Gamma_2 = 1/\sqrt{1 - \beta^2} \) that of the shocked gas as seen by the observer in the medium at rest. It is easy to show that

\[ \Gamma_2 = \left[ \frac{(\epsilon_1 + p_2)(\epsilon_2 + p_1)}{(\epsilon_1 + p_1)(\epsilon_2 + p_2)} \right]^{1/2} \] (B.7)

Instead of the energy behind the shock it is usually more convenient to consider this, or \( \Gamma_s \), as the main parameter of the shock.

Let us now consider some special cases. First, assume that the shock is propagating into a cold medium with density \( \rho_1 = n_1 m_u \) and negligible

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pressure, so that $\epsilon_1 = n_1 m_u c^2$ and $p_1 = 0$. Further, we assume that the gas behind the shock is relativistic, so that $p_2 \approx \epsilon_2/3$. We then obtain from Eq. (B.7)

$$\Gamma_2 = \frac{1}{2} \left( \frac{\epsilon_2}{n_1 m_u c^2} + 3 \right)^{1/2}. \quad \text{(B.8)}$$

The velocities of the preshock and postshock gas in the shock frame are Eq. (B.5)

$$\beta_1 \approx 1 - \frac{n_1 m_u c^2}{\epsilon_2} \quad \text{(B.9)}$$

$$\beta_2 \approx \frac{1}{3} \left( 1 + 2 \frac{n_1 m_u c^2}{\epsilon_2} \right) \quad \text{(B.10)}$$

showing that the gas is flowing in with nearly the speed of light, while the postshock gas flows away with $\beta_2 \sim 1/3$.

Using Eq. (B.10) it is easy to show that therefore

$$\Gamma_s = \gamma_1 \approx \left( \frac{\epsilon_2}{2 n_1 m_u c^2} \right)^{1/2}. \quad \text{(B.11)}$$

Therefore, the shock Lorentz factor is a factor of $2^{1/2}$ larger than that of the postshock gas as seen by the observer at rest,

$$\Gamma_s \approx 2^{1/2} \Gamma_2 \quad \text{(B.12)}$$

Using Eq. (B.10) in the first of the jump conditions in Eq. (B.3) we obtain

$$\frac{n_2}{n_1} \approx 2 \left( \frac{\epsilon_2}{n_1 m_u c^2} \right)^{1/2} \approx 4 \Gamma_2 \quad \text{(B.13)}$$

In contrast to a non-relativistic shock we can therefore get arbitrarily large compressions behind the shock. This relation, together with Eq. (B.11), also shows that most of the incoming kinetic energy of the particles in the shock frame, $\sim \Gamma_s m_u c^2$, is converted to thermal, internal energy.

Equations (B.8), (B.12), and (B.13) form the basic relations describing the properties of the shock wave as function of its Lorentz factor and preshock density, or alternatively the energy density behind the shock and the preshock density.

The shape and flux of the synchrotron emission is determined by the magnetic field and density of relativistic particles behind the shock. Lacking a fundamental theory, one is usually assuming that the energy densities of these scale with the thermal energy density behind the shock. Therefore

$$\frac{B^2}{8\pi} = \varepsilon_B \epsilon_2 \approx 4 \varepsilon_B \Gamma_2^2 n_1 m_u c^2 \quad \text{(B.14)}$$
For a power law spectrum $n_{rel} = C \gamma^{-p}$ the ratio of the energy density and number density is given by

$$\frac{u_{rel}}{m_c^2 n_{rel}} = \frac{(p-1)}{(p-2)} m_e c^2 \gamma_{\text{min}}$$

(B.16)

where $\gamma_{\text{min}}$ is the minimum energy, and we assume that $p > 2$, so that we can omit the upper limit to $\gamma$.

To determine $\gamma_{\text{min}}$ we divide Eq. (B.15) by Eq. (B.13) to obtain

$$\frac{u_{rel}}{n_{2}} = \varepsilon_e m_p c^2 \Gamma_2 .$$

(B.17)

If we now assume that $n_{rel} \approx n_2$ we get

$$\gamma_{\text{min}} = \varepsilon_e \frac{m_p (p-2)}{m_e (p-1)} \Gamma_2 .$$

(B.18)

Note that this is based on the assumptions that the relativistic particle density scales as the postshock energy density, and that number of relativistic particles also scale with the postshock particle density. Although, a likely situation, this has to be justified by detailed simulations of collisionless shocks. There is currently substantial work going on in this area.

**C Relativistic blast waves**

Immediately after the explosion the hot ejecta is basically a ball of hot photons, with a very small baryon load $M_0$ (Eq. 12.6). ..... It therefore expands with $\Gamma \propto R$, until $\Gamma = \Gamma_0 \approx E/M_0 c^2$, after which it expands with constant $\Gamma$ until it has swept up an energy comparable to the initial. The shock condition Eq. (B.8) shows that in the rest frame of the shocked fluid the energy is $E \approx \Gamma mc^2$, and therefore in the rest frame of the observer $E_{\text{obs}} \approx \Gamma^2 mc^2$. The energy swept up is therefore comparable to the initial thermal energy when the mass of the swept up ejecta is $m \approx M_0/\Gamma_0$. At this point the ejecta will start to slow down.

In the same way as the Sedov solution plays a central role for the dynamics of the interaction of the SN with its environment, one can find relativistic generalizations of these, which describes the slowing down of the blast wave, and the conversion of the kinetic energy to thermal energy behind the blast wave.

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From Eq. (B.11) and Eq. (B.12), we find that the energy behind the shock in the comoving frame is
\[ \epsilon_2 = 4\Gamma_2^2 m_u c^2 n_1 \] (C.1)

Because the post-shock gas is relativistic, \( p_2 = \epsilon_2/3 \). To transform \( \epsilon_2 \) and \( p_2 \) to the observer frame we use
\[ \epsilon_{\text{obs}} = \Gamma_2^2 (\epsilon_c + \beta^2 p_c) = \Gamma_2^2 \epsilon_c (1 + \frac{1}{3} \beta^2) = (4\Gamma_2^2 - 1) \frac{\epsilon_c}{3} \] (C.2)

With \( \epsilon_c = \epsilon_2 \) from Eq. (C.1), we get for large \( \Gamma_2 \)’s
\[ \epsilon_{\text{obs}} \approx \frac{16}{3} \Gamma_2^4 m_u c^2 n_1 \] (C.3)

The total energy in the observer frame is therefore
\[ E_{\text{obs}} \approx \frac{64\pi}{3} \Gamma_2^4 m_u c^2 n_1 R^2 \Delta R_{\text{obs}} \] (C.4)

where \( \Delta R_{\text{obs}} \) is the shock thickness as seen in the observer frame. To estimate \( \Delta R_{\text{obs}} \) we use the conservation of mass. For constant external density the total mass swept up by the shock is \( 4\pi R_1^3 m_u n_1/3 \). This should be equal to the mass in the shell, \( 4\pi R^2 \Delta R_{\text{obs}} n_{2, \text{obs}} m_u \). The density behind the shock in the observer frame is \( n_{2, \text{obs}} = \Gamma_2 n_{2, c} \). The comoving density is given in terms of the pre-shock density by Eq. (B.13). Therefore,
\[ n_{2, \text{obs}} = 4 \Gamma_2^2 n_1 \] (C.5)

and
\[ \Delta R_{\text{obs}} \approx \frac{R}{12\Gamma_2^2} \] (C.6)

In the comoving frame of the shocked gas \( \Delta R/R = 1/(12\Gamma_2) \). The shell is therefore even in the comoving frame extremely thin, due to the large compression behind the shock.

Inserting this in Eq. (C.4) we get
\[ \Gamma_2 \approx \left( \frac{9E}{16\pi n_1 m_u c^2} \right)^{1/2} R^{-3/2} \] (C.7)

This is the Blandford – McKee solution, and describes together with Eqns. (B.12), (C.1), and (C.5) the physical conditions of the blast wave as function of its radius.
To transform this into a relation of time as measured by the observer we use Eq. (12.4), $dt_{\text{obs}} = dt_{\text{em}}/2\Gamma_s^2$, and that $dR = c\beta dt_{\text{em}} \approx cdt_{\text{em}}$, so that

$$t_{\text{obs}} = \int \frac{dt_{\text{em}}}{2\Gamma_s^2} = \int_0^R \frac{dR'}{2c\Gamma_s(R')^2}. \quad \text{(C.8)}$$

Therefore, if $\Gamma_s \propto R^{-\alpha}$, we get

$$R = 2(1 + 2\alpha)\Gamma_s(R)^2ct_{\text{obs}} \quad \text{(C.9)}$$

so that with $\alpha = 3/2$ we get $R = 8\Gamma_s^2ct_{\text{obs}}$. Using this in Eq. (C.7) we get

$$\Gamma_s \approx 0.4 \left(\frac{E}{n_1 m_u c^5}\right)^{1/8} t_{\text{obs}}^{-3/8} \quad \text{(C.10)}$$

or with parameters considered to be typical for a GRB

$$\Gamma_s \approx 4.4 \left(\frac{E}{10^{52} \text{ ergs}}\right)^{1/8} \left(\frac{n_1}{1 \text{ cm}^{-3}}\right)^{-1/8} \left(\frac{t_{\text{obs}}}{\text{days}}\right)^{-3/8} \quad \text{(C.11)}$$

and

$$R \approx \left(\frac{9E}{2\pi n_1 m_u c}\right)^{1/8} t_{\text{obs}}^{1/4} \quad \text{(C.12)}$$

or

$$R \approx 4 \times 10^{17} \left(\frac{E}{10^{52} \text{ ergs}}\right)^{1/8} \left(\frac{n_1}{1 \text{ cm}^{-3}}\right)^{-1/8} \left(\frac{t_{\text{obs}}}{\text{days}}\right)^{1/4} \quad \text{(C.13)}$$

The derivation above, as well as the total energy used here, assume a spherical expansion. As we will see, it is, however, likely that the outflow is in the form of two narrow jets. As long as the jet angle is much larger than $\Gamma_s^{-1}$ this is, however, of minor importance, except for the fact that the energy going into these equations is the equivalent isotropic energy, $E_{\text{iso}}$. The real total energy is only $\Omega E_{\text{iso}}/2\pi$, where $\Omega$ is the solid angle of each of the two jets.