Astrophysical Gasdynamics Exercises

① Mean freez path: $\lambda = \frac{1}{\sqrt{2}\, n \pi a^2} \sim \frac{1}{n\pi a^2}$

$n = 10^{-4}\ \mathrm{cm^{-3}} \rightarrow \lambda = 1.56 \times 10^{17}\ \mathrm{cm} = 5.07\ \mathrm{pc} = 1.04 \times 10^{6}\ \mathrm{AU}$

$n = 1\ \mathrm{cm^{-3}} \rightarrow \lambda = 1.56 \times 10^{13}\ \mathrm{m} = \cancel{0.0507\ \mathrm{pc}}\ 5.07\times10^{-4}\ \mathrm{pc} = 1.04 \times 10^{2}\ \mathrm{AU}$

$n = 10^{4}\ \mathrm{cm^{-3}} \rightarrow \lambda = 1.56 \times 10^{9}\ \mathrm{m} = \cancel{0.0000507\ \mathrm{pc}}\ 5.07\times10^{-8}\ \mathrm{pc} = 0.0104\ \mathrm{AU}$

→ mostly ok. Low density gas on small scales problematic, but there the gas is usually ionized, and collisions are due to Coulomb forces, and are also influenced by the presence of magnetic fields.

② See notes

③ ⓐ $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u} = 0$

$$\frac{D\rho}{Dt} - \vec{u}\cdot\vec{\nabla}\rho + \vec{\nabla}\cdot\rho\vec{u} = \frac{D\rho}{Dt} + \rho\,\vec{\nabla}\cdot\vec{u} = 0$$

$\left(\frac{D\rho}{Dt} = 0 \rightarrow \vec{\nabla}\cdot\vec{u} = 0\right.$, incompressible flow)

divergence free

ⓑ $\frac{\partial \rho u_i}{\partial t} + \vec{\nabla}\cdot(\rho u_i \vec{u}) = -\vec{\nabla}_i p + \rho g_i$

$$\frac{D\rho u_i}{Dt} - \vec{u}\cdot\vec{\nabla}(\rho u_i) + \vec{\nabla}\cdot(\rho u_i\vec{u}) = \frac{D\rho u_i}{Dt} + \rho u_i\,\vec{\nabla}\cdot\vec{u} = \vec{\nabla}_i p + \rho g_i$$

ⓒ $\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E+p)\vec{u} = -\rho Q_{cool} + \rho\vec{u}\cdot\vec{g}$

$$\frac{DE}{Dt} - \vec{u}\cdot\vec{\nabla}E + \vec{\nabla}\cdot(E\vec{u}) + \vec{\nabla}\cdot(p\vec{u}) = \frac{DE}{Dt} + E\vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot(p\vec{u}) = -\rho\dot{Q}_{cool} + \rho\vec{u}\cdot\vec{g}$$

(d) $\frac{\partial \rho u_i}{\partial t} + \vec{\nabla}\cdot(\rho u_i \vec{u}) = -\nabla_i p + \rho g_i$

$$\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} \quad \hookrightarrow u_i(-\vec{\nabla}\cdot(\rho\vec{u}))$$

$$\rho \frac{\partial u_i}{\partial t} - u_i \vec{\nabla}\cdot(\rho\vec{u}) + \vec{\nabla}\cdot(\rho u_i \vec{u}) = \rho\frac{\partial u_i}{\partial t} + \rho\vec{u}\cdot\vec{\nabla}u_i = -\nabla_i p + \rho g_i$$

$$\rightarrow \frac{\partial u_i}{\partial t} + \vec{u}\cdot\vec{\nabla}u_i = -\nabla_i p + \rho g_i$$

$$\frac{Du_i}{Dt} - \vec{u}\cdot\vec{\nabla}u_i + \vec{u}\cdot\vec{\nabla}u_i = \frac{Du_i}{Dt} = -\nabla_i p + g_i$$

or $\frac{D\vec{u}}{Dt} = -\vec{\nabla}p + \vec{g}$

(4) (a) $\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E+p)\vec{u} = -\rho\dot{Q}_{cool} + \rho\vec{u}\cdot\vec{g}$

$$E = \rho\varepsilon + \frac{1}{2}\rho u^2$$

$$\frac{\partial \frac{1}{2}\rho u_i^2}{\partial t} = u_i\frac{\partial \rho u_i}{\partial t} - \frac{1}{2}u_i^2\frac{\partial \rho}{\partial t}$$

$$= u_i\left[-\vec{\nabla}\cdot(\rho u_i\vec{u}) - \vec{\nabla}_i p + \rho g_i\right] - \frac{1}{2}u_i^2\left[-\vec{\nabla}\cdot(\rho\vec{u})\right]$$

$$= -u_i\vec{\nabla}\cdot(\rho u_i\vec{u}) + \frac{1}{2}u_i^2\vec{\nabla}\cdot(\rho\vec{u}) - u_i\nabla_i p + \rho g_i u_i$$

$$= -\vec{\nabla}\cdot\left(\frac{1}{2}\rho u_i^2\vec{u}\right) - u_i\nabla_i p + \rho g_i u_i$$

Valid for all i $\rightarrow$ $u^2 = \sum_{i=1}^{3} u_i^2$

$$\boxed{\frac{\partial \frac{1}{2}\rho u^2}{\partial t} + \vec{\nabla}\cdot\left(\frac{1}{2}\rho u^2\vec{u}\right) = -\vec{u}\cdot\vec{\nabla}p + \rho\vec{u}\cdot\vec{g}}$$

kinetic energy equation (aka work equation)

$$\frac{\partial \rho \varepsilon}{\partial t} = \frac{\partial E}{\partial t} - \frac{\partial \frac{1}{2}\rho u^2}{\partial t}$$

$$= -\vec{\nabla}\cdot(\rho\varepsilon + \tfrac{1}{2}\rho u^2 + p)\cdot\vec{u} - \rho\dot{Q}_{cool} + \rho\vec{u}\cdot\vec{g}$$
$$+ \vec{\nabla}\cdot(\tfrac{1}{2}\rho u^2 \vec{u}) + \vec{u}\cdot\vec{\nabla}p - \rho\vec{u}\cdot\vec{g}$$

$$= -\vec{\nabla}\cdot(\rho\varepsilon\vec{u}) - \vec{\nabla}\cdot(p\vec{u}) + \vec{u}\cdot\vec{\nabla}p - \rho\dot{Q}_{cool}$$
$$= -\vec{\nabla}\cdot(\rho\varepsilon\vec{u}) - p\vec{\nabla}\cdot\vec{u} - \rho\dot{Q}_{cool}$$

$$\Rightarrow \frac{\partial \rho\varepsilon}{\partial t} + \vec{\nabla}\cdot(\rho\varepsilon\vec{u}) + p\vec{\nabla}\cdot\vec{u} = -\rho\dot{Q}_{cool}$$

b) $S = C_v \log(p\rho^{-\gamma}) + \text{const}$

$\rightarrow$ varies with $p\rho^{-\gamma}$

$$\frac{\partial p\rho^{-\gamma}}{\partial t} = \rho^{-\gamma}\frac{\partial p}{\partial t} + p\frac{\partial \rho^{-\gamma}}{\partial t}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\partial \rho\varepsilon(\gamma-1)}{\partial t} \qquad -\gamma\rho^{-\gamma-1}\frac{\partial\rho}{\partial t}$$

$$= \qquad\qquad \downarrow$$

$$-(\gamma-1)[\vec{\nabla}\cdot(\rho\varepsilon\vec{u}) - p\vec{\nabla}\cdot\vec{u}) \qquad -\gamma\rho^{-\gamma-1}\left[-\vec{\nabla}\cdot\rho\vec{u}\right]$$

$$\Rightarrow -\rho^{-\gamma}\vec{\nabla}\cdot(p\vec{u}) - \rho^{-\gamma}(\gamma-1)\underline{p\vec{\nabla}\cdot\vec{u}} + \gamma p\rho^{-\gamma-1}\left[\vec{u}\cdot\vec{\nabla}\rho + \underline{\rho\vec{\nabla}\cdot\vec{u}}\right] =$$

$$= -\rho^{-\gamma}(\vec{\nabla}\cdot(p\vec{u}) - p\vec{\nabla}\cdot\vec{u}) + \gamma p\rho^{-\gamma-1}\vec{u}\cdot\vec{\nabla}\rho =$$

$$= -\rho^{-\gamma}\vec{u}\cdot\vec{\nabla}p - p\vec{u}\cdot\vec{\nabla}\rho^{-\gamma} = -\vec{u}\cdot\vec{\nabla}p\rho^{-\gamma}$$

$$\Rightarrow \frac{\partial p\rho^{-\gamma}}{\partial t} + \vec{u}\cdot\vec{\nabla}p\rho^{-\gamma} = \frac{Dp\rho^{-\gamma}}{Dt} = 0 \quad \text{if } \dot{Q}_{cool} = 0$$

Can solve for internal energy when entropy is conserved $\rightarrow$ no irreversible processes (heating/cooling, shocks)

(c) $P = k\rho^{\gamma}$ $\quad p = \rho \frac{R_*}{\mu} T \quad \rightarrow \quad p \propto \rho \quad \rightarrow \gamma = 1$

$$k = \frac{R_*}{\mu} T$$

Energy equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$

since $p = k\rho$

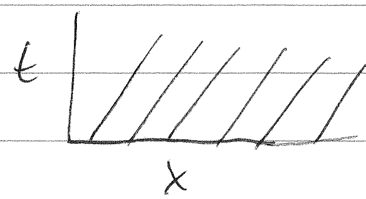
(5) a) $\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x}$

$$\rho(x,t) = \rho(x - u_0 t, t=0)$$

$$\frac{\partial \rho(x-u_0t)}{\partial t} + \frac{\partial \rho(x-u_0t)}{\partial x} = \frac{\partial \rho}{\partial(x-u_0t)} \frac{\partial x-u_0t}{\partial t} + \frac{\partial \rho}{\partial x-u_0t} \frac{\partial x-u_0t}{\partial x}$$

$$= \frac{\partial \rho}{\partial(x-u_0t)} \cdot -u_0 + u_0 \frac{\partial \rho}{\partial(x-u_0t)} = 0 \quad \text{Q.E.D.}$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u_0}{\partial x} = 0$$



b) Argument:

ρ changes along flow lines: the flow is expanding, since $u(x) = x$, so larger velocities for larger x. The same amount of mass needs to fill a larger space $\rightarrow$ ρ is going down.

For $x<0$, $v<0$
$x=0$, $v=0$ $\rightarrow$ $x<0$ and $x>0$ are separated
$x>0$, $v>0$

Lagrangian $\frac{D\rho}{Dt} = -\rho\frac{\partial u}{\partial x} = -\rho \rightarrow \rho(t) = \rho_0 e^{-t}$

Particle paths: $\frac{dt}{dx} = \frac{1}{x} \rightarrow t = \log x + C$

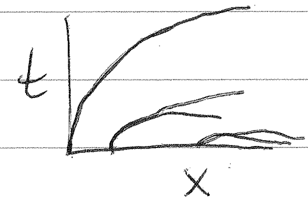
at t_0, x_0: $t_0 = \log x_0 + C \rightarrow C = t_0 - \log x_0$

$\rightarrow t - t_0 = \log x/x_0 \rightarrow e^{t-t_0} = x/x_0$

$x_0 = xe^{-t/t_0}$

for $t_0 = 0$, $x_0 = xe^{-t}$

$\rightarrow \rho(x,t) = \rho_0(xe^{-t})e^{-t}$



(6) Particles without a f_{MB} (approximately) $\rightarrow$ fluid equations do not hold.

Note: Careful with unusual symbols here

The momentum equation can be rewritten as

$$\frac{\partial v}{\partial t} + v\frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{dp}{dr} = f_r$$

(7) Now let us consider the atmosphere of a star. We assume that the temperature is constant, so

$$p = \frac{k_B \rho T}{m} = \rho \mathcal{R} T$$

only depends on ρ. This eliminates the need to solve for the energy equation.

The force on the gas is the gravity of the star:

$$f = -\frac{G m_*}{r^2}$$

So, the hydrostatic ($v=0$, $\frac{\partial}{\partial t}=0$) solution should follow from

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{G m_*}{r^2}$$

If we define $a = \sqrt{p/\rho} = \sqrt{\frac{k_B T}{m}}$ as the isothermal sound speed the equation becomes

$$\frac{r^2}{\rho}\frac{\partial \rho}{\partial r} = -\frac{G m_*}{a^2}$$

with the solution

$$\rho(r) = \rho_0 \exp\left(-\frac{G m}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right)$$

The exponential atmosphere.

If $r = r_0 + \Delta r$ and $\Delta r << r_0$

$$\rho(r) = \rho_0 \exp\left(-\underbrace{\boxed{\frac{G m}{r_0^2}}}_{g}\frac{m \Delta r}{k_B T}\right)$$

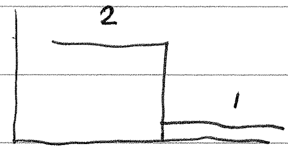
$$= \rho_0 \exp(-\Delta r / H) \qquad H = \frac{k_B T}{m g}$$

↳ scale height

(4.32)

(8) (a) $\mathcal{M}_1 = 3$

$\rho_1 = 5/3, \quad u_1' = 10, \quad p_1 = 1$



$c_1 = \sqrt{\frac{\gamma p_1}{\rho}} = 1$

Since $\mathcal{M}_1 = \left|\frac{u_1}{c_1}\right| \rightarrow |u_1| = 3$ $\qquad u_1 = u_1' - V_{sh}$

$\Rightarrow V_{sh} = u_1' - u_1$

If $u_1 = +3 \rightarrow V_{sh} = 7$

but in shock frame the material does <u>not</u> flow <u>into</u> the shock ($u_1 > 0$)

$\rightarrow$ wrong solution

$\Rightarrow u_1 = -3$ and $V_{sh} = 13$

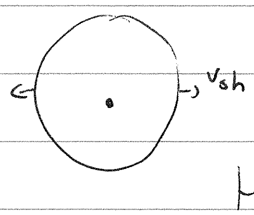
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}^2}{2+(\gamma-1)\mathcal{M}^2} = 3 \rightarrow \rho_2 = 5$$

$$\frac{u_2}{u_1} = \frac{1}{3} \rightarrow u_2 = -1 \rightarrow u_2' = -1 + V_{sh} = 12$$

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}^2 - (\gamma-1)}{\gamma+1} = 11 \rightarrow p_2 = 11$$

$\Rightarrow \rho_2 = 5, \quad u_2' = 12, \quad p_2 = 11 \qquad V_{sh} = 13$

(b)



$$\Rightarrow \frac{D\Delta\theta}{\Delta t} = V_{sh} \rightarrow D = V_{sh}\left(\frac{\Delta t}{\Delta\theta}\right)$$

However, spectroscopy gives us u_2'. If we use u_2' instead of V_{sh} we find a wrong distance $\tilde{D} = u_2'\left(\frac{\Delta t}{\Delta\theta}\right)$

$\cancel{\frac{D}{\tilde{D}} = \frac{V_{sh}}{u_2'}}$ Relative error made: $\frac{\tilde{D} - D}{D} = \frac{u_2' - V_{sh}}{V_{sh}}$

$= \frac{12-13}{13} = -7.7\%$ $\rightarrow$ we underestimate the distance by $\sim 8\%$

(9) $F_2 - F_1 = V_{sh}(W_2 - W_1)$ in observer's frame

For $W = \rho$ $\rightarrow$ $\rho_2 u_2' - \rho_1 u_1' = V_{sh}(\rho_2 - \rho_1)$

since $u' = u + V_{sh}$

$$\rho_2(u_2 + V_{sh}) - \rho_1(u_1 + V_{sh}) =$$

$$\underbrace{\rho_2 u_2 - \rho_1 u_1}_{0} + \rho_2 V_{sh} - \rho_1 V_{sh} = V_{sh}(\rho_2 - \rho_1) \quad \text{QED}$$

For $W = \rho u$ $\rightarrow$ $\rho_2 u_2'^2 + p_2 - (\rho_1 u_1'^2 + p_1) =$

$$\rho_2 (u_2 + V_{sh})^2 + p_2 - \rho_1 (u_1 + V_{sh})^2 + p_1 =$$

$$\underbrace{(\rho_2 u_2^2 + p_2) - (\rho_1 u_1^2 + p_1)}_{0} + 2\rho_2 u_2 V_{sh} + \rho_2 V_{sh}^2$$

$$-2\rho_1 u_1 V_{sh} - \rho_2 V_{sh}^2 =$$

$$V_{sh}\left(\rho_2(u_2 + V_{sh}) - \rho_1(u_1 + V_{sh}) + \underbrace{\rho_2 u_2 - \rho_1 u_1}_{0}\right)$$

$$= V_{sh}(\rho_2 u_2' - \rho_1 u_1')$$

For $W = E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}$ $\rightarrow$ $\left[\left(\frac{1}{2}\rho(u_2 + V_{sh})^2 + \frac{p_2}{\gamma-1} + p_2\right)\right](u_2 + V_{sh}) - [\;\;]$

$$= \underbrace{\left(\frac{1}{2}\rho_2(u_2 + V_{sh})^2 + \frac{p_2}{\gamma - 1}\right)}_{W_2} V_{sh} + p_2 V_{sh} + \left(\frac{1}{2}\rho_2(u_2 + V_{sh})^2 + \frac{\gamma p_2}{\gamma - 1}\right)u_2$$

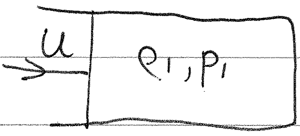
$$-[\;\;]$$

$$\rightarrow \underbrace{p_2 V_{sh} + \left(\frac{1}{2}\rho u_2^2 + \frac{\gamma p_2}{\gamma - 1}\right)u_2}_{\text{R-H term}} + \rho u_2^2 V_{sh} + \frac{1}{2}\rho V_{sh}^2 u_2$$

$$\rightarrow V_{sh}\Big(\underbrace{[\rho_2 u_2^2 + p_2]}_{\text{R-H term}} + \tfrac{1}{2}V_{sh}\underbrace{[\rho u]}_{\text{R-H term}}\Big)$$

All R-H term cancel, so the only remaining terms are : $W_2 V_{sh} - W_1 V_{sh}$ Q.E.D.

(10)



a) $C_s^2 = \gamma p_1/\rho_1$

$\rightarrow u^2 > c_s^2$: shockwave

b)

u_s

ρ_2	ρ_1
$u_2' = u$	$u_1' = 0$
p_2	$p_1 = 0$

inthe shock frame

ρ_2	ρ_1
$u_2 = u - u_s$	$u_1 = -u_s$
p_2	p_1

$$M_1^2 = \left|\frac{u_1^2}{c_1^2}\right| = \frac{u_s^2}{\gamma p_1/\rho_1} \rightarrow \frac{p_2}{p_1} = \frac{2}{\gamma+1}\frac{\gamma u_s^2}{\gamma p_1/\rho_1} - \frac{\gamma-1}{\gamma+1} \quad (1)$$

Also $$\left.\begin{array}{l} p_1 + \rho_1 u_1^2 = p_1 + \rho_1 u_s^2 = p_2 + \rho_2 (u - u_s)^2 \\ \rho_2(u - u_s) = -\rho_1 u_s \end{array}\right\} \Rightarrow$$

$$p_2 = p_1 + \rho_1 u_s u \quad (2)$$

$$(1),(2):\ 1 + \frac{\rho_1}{p_1} u_s u = \frac{2\gamma u_s^2}{(\gamma+1)\gamma p_1/\rho_1} - \frac{\gamma-1}{\gamma+1}$$

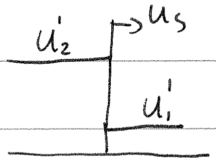
$$\Rightarrow u_s^2 - \frac{(\gamma+1)u}{2} u_s - \frac{\gamma P_1}{\rho_1} = 0$$

$$\Rightarrow u_s = (\gamma+1)\frac{u}{4} + \sqrt{\frac{(\gamma+1)^2 u^2}{16} + \frac{\gamma P_1}{\rho_1}}$$

For large u, $u^2 \gg \gamma P_1/\rho_1$

$$\rightarrow u_s = (\gamma+1)\frac{u}{2} = \frac{4}{3}u$$

11 a



$$u_1 = u_1' - u_s \qquad u_1' = 0$$
$$u_2 = u_2' - u_s$$

$$c_1 = \sqrt{\frac{\gamma P_1}{\rho_1}} = \sqrt{\frac{\gamma R}{\mu}} = 1.17 \text{ km s}^{-1} \quad (\mu = 1.07 \text{ for } H)$$

$$M_1 = \frac{|u_1|}{c_1} = 4 \rightarrow |u_1| = 4 \cdot 1.17 \text{ km s}^{-1}$$
$$u_1 = -4.69 \text{ km s}^{-1}$$ (pre-shock gas has to flow into the shock)

$$u_1' = 0 \rightarrow u_{s,1} = 4.69 \text{ km s}^{-1}$$

$$\frac{\rho_2}{\rho_1} = \frac{n_2}{n_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} = \frac{u_1}{u_2} = 3.37$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{(\gamma+1)} = 19.75$$

$$u_2' = u_2 + u_{s,1} = \frac{u_1}{3.4} + u_{s,1} = -\frac{4.69}{3.37} + 4.69 = 3.30 \text{ km s}^{-1}$$

$$n_2 = 3.37 \text{ cm}^{-3}$$

$$\frac{T_2}{T_1} = \frac{P_2/\rho_2}{P_1/\rho_1} = 5.86 \rightarrow T_2 = 586\,K$$

b) $\mathcal{M}_2 = 2$ pre-shock: $\underline{2}$, postshock: $\underline{3}$

$$c_2 = \sqrt{\frac{R T_2}{\mu}} = 2.84\ km\,s^{-1}$$

$$\rightarrow |u_2| = \mathcal{M} c_2 = 5.68\ km\,s^{-1}$$

$$u_2 = -5.68\ km\,s^{-1}$$

$$= u_2' - u_{s,2} = 3.30 - u_s$$

$$\rightarrow u_{s,2} = 8.98\ km\,s^{-1}$$

$$(> u_{s,1})$$

$$\frac{\rho_3}{\rho_2} = \frac{n_3}{n_2} = 2.29 \rightarrow n_3 = 7.72\ cm^{-3}$$

$$\frac{P_3}{P_2} = 4.75 \qquad \rightarrow T_3 = \frac{4.75}{2.29} T_2 = 1216\,K$$

$$u_3 = \frac{u_2}{2.29} = -2.48\ km\,s^{-1} \rightarrow u_3' = 8.98 - 2.48$$

$$= 6.50\ km\,s^{-1}$$

c) $\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1} \qquad \frac{P_2}{P_1} = \frac{2\gamma \mathcal{M}^2}{\gamma+1}$

d) As above, but without the numbers

$$u_1 = -\mathcal{M}_1 c_1 \rightarrow u_{s,1} = \mathcal{M}_1 c_1$$

($u_1' = 0$ can always be our lab frame)

$$u_2 = -\frac{\gamma-1}{\gamma+1}\mathcal{M}_1 c_1 \qquad u_2' = u_2 + u_{s,1} = \frac{2}{\gamma+1}\mathcal{M}_1 c_1$$

$$c_2^2 = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} c_1^2 = \frac{2\gamma M^2}{\gamma+1} \frac{\gamma-1}{\gamma+1} c_1^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 c_1^2$$

In frame of shock 2:

$$u_2 = -M_2 c_2 = -M_2 \sqrt{\frac{2\gamma(\gamma-1) M_1^2}{(\gamma+1)^2}}\, c_1$$

$$u_2' = u_2 + u_{s,2}$$

$$\rightarrow u_{s,2} = u_2' - u_2 = \frac{2}{\gamma+1} M_1 c_1 + M_1 M_2 \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1} c_1$$

$$= M_1 c_1 \left[\frac{2}{\gamma+1} + M_2 \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1} \right]$$

$$= u_{s,1} \left[\frac{2}{\gamma+1} + M_2 \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1} \right]$$

$$\rightarrow \frac{u_{s,2}}{u_{s,1}} = \frac{2}{\gamma+1} + M_2 \frac{\sqrt{2\gamma(\gamma-1)}}{(\gamma+1)} > \frac{2 + \sqrt{2\gamma(\gamma-1)}}{(\gamma+1)} > 1$$

Q.E.D.

(12) If the shock position is only a function of E, ρ_1 and t then

(a) $$r \propto E^{\alpha} \rho_1^{\beta} t^{\gamma}$$

$[E] = J = kg\ m^2 s^{-2}$

$[\rho_1] = kg\ m^{-3}$

$[t] = s$

$$\Rightarrow \left.\begin{array}{ll} (\alpha+\beta)\ kg & \to 0 \\ (2\alpha - 3\beta)\ m & \to 1 \\ (-2\alpha + \gamma)\ s & \to 0 \end{array}\right\} \qquad \alpha = -\beta \Rightarrow -5\beta = 1 \Rightarrow \beta = -1/5$$

$\alpha = 1/5$

$\gamma = 2/5$

$$r \propto E^{1/5} \rho_1^{-1/5} t^{2/5}$$

(b) $$r \propto \dot{E}^{\alpha} \rho_1^{\beta} t^{\gamma}$$

$$\left.\begin{array}{ll} (\alpha+\beta)\ kg & \to 0 \\ (2\alpha - 3\beta)\ m & \to 1 \\ (-3\alpha + \gamma)\ s & \to 0 \end{array}\right\} \qquad \begin{array}{l} \alpha = 1/5 \\ \beta = -1/5 \\ \gamma = 3/5 \end{array}$$

(c) $[L] = J\ s^{-1}$ energy per second

$\frac{1}{2}\dot{m} u_w^2$ is also energy per second: kinetic energy in the wind.

(13) (a) Adiabatic, meant polytropic : $P = K\rho^{\gamma}$

$$h = \int \frac{dP}{\rho} = \int K\gamma \frac{\rho^{\gamma-1}}{\rho} d\rho = \int \gamma K \rho^{\gamma-2} d\rho = \int \frac{\gamma K}{\gamma - 1} d\rho^{\gamma-1}$$

$$= \frac{\gamma}{\gamma-1} P/\rho$$

Energy equation: $\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E+p)\vec{u} = \rho\vec{u}\cdot\vec{g}$

If we use $E_{tot} = \frac{1}{2}\rho u^2 + \rho\varepsilon + \rho\Psi = E + \rho\Psi$

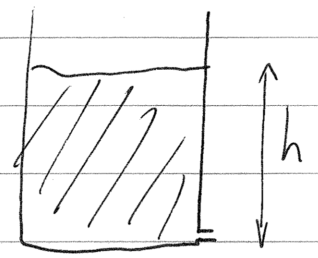
we get $\frac{\partial E_{tot}}{\partial t} + \vec{\nabla}\cdot(E_{tot}+p)\vec{u} = \rho\frac{\partial \Psi}{\partial t}$

For steady flow $\partial/\partial t = 0 \rightarrow \vec{\nabla}\cdot(E_{tot}+p)\vec{u} = 0$

$$\vec{\nabla}\cdot(E_{tot}+p)\vec{u} = \frac{(E_{tot}+p)}{\rho}\underbrace{\vec{\nabla}\cdot\rho\vec{u}}_{\substack{=0 \\ \text{for steady} \\ \text{flow}}} + \rho\vec{u}\cdot\vec{\nabla}\frac{(E_{tot}+p)}{\rho}$$

$$\Rightarrow \frac{E_{tot}+p}{\rho} = \frac{1}{2}u^2 + \frac{\gamma p/\rho}{\gamma-1} + \Psi = \text{constant along a streamline}$$

(b)



surface: $\frac{p_0}{\rho} + gh$

hole: $\frac{p_0}{\rho} + \frac{1}{2}u^2$

It's a liquid, so $\rho = \rho$.
p_0 is the atmospheric pressure
$\rightarrow \frac{1}{2}u^2 = gh \rightarrow u = \sqrt{2gh}$

(14) a) See lecture

b) $H = \frac{1}{2}u^2 + \frac{\gamma p/\rho}{\gamma-1} = \frac{1}{2}u^2 + \frac{c_s^2}{\gamma-1}$

At the start of the nozzle: $u \approx 0$, $c_s = c_0$

So $H = \frac{c_0^2}{\gamma - 1}$

At A_{min}: $u = c_*$ $\rightarrow$ $\frac{c_0^2}{\gamma-1} = \frac{1}{2}c_*^2 + \frac{c_*^2}{\gamma-1}$

$\Rightarrow c_* = c_0\sqrt{2/(\gamma+1)}$

c) No shocks or outside perturbations, radiative losses, etc $\rightarrow$ entropy conserved.

At start of nozzle: $p = p_0$, $\rho = \rho_0$ $\rightarrow$ $p_0 \rho_0^{-\gamma}$ conserved.

At A_{min}: $\rho = \rho_*$, $p = p_*$ $\quad p_* \rho_*^{-\gamma} = p_0 \rho_0^{-\gamma}$

$$\Downarrow$$

$$c_*^2 \rho_*^{1-\gamma} = c_0^2 \rho_0^{1-\gamma}$$

$$\Downarrow$$

$$\rho_* = \rho_0 \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1}$$

d) $\dot{m} = \rho u A$ everywhere

$$\dot{m} = \rho_* u_* A_{min} = \rho_0 \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1} c_* A_{min}$$

$$= \rho_0 \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1} c_0 \left(\frac{2}{\gamma+1}\right)^{1/2} A_{min}$$

$$= \sqrt{\gamma p_0 \rho_0} \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(2\gamma-2)} A_{min} \qquad \frac{1}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}$$

$$\rightarrow A_{min} = \frac{Q}{\sqrt{\gamma p_0 \rho_0}} \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(2-2\gamma)}$$

(15) a) Strong shock: $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} = 4$

b) Along streamline, $H = \frac{1}{2}u^2 + \frac{\gamma P/\rho}{\gamma-1}$ is constant (Bernoulli)

Across a shock $\frac{1}{2}u^2 + \frac{\gamma P/\rho}{\gamma-1}$ is constant (Rankine-Hugoniot)

c) $\mathcal{E} = \frac{1}{2}(u_j - u_h)^2 + 5P_j/2\rho_j$ = constant

In the jet $|u_j - u_h| \gg \sqrt{\frac{5P_j}{2\rho_j}}$

$\rightarrow \mathcal{E} = \frac{1}{2}|u_j - u_h|^2$

At the stagnation point: $u = 0$

$\rho_{sj} = 4\rho_j$

$\mathcal{E} = \frac{1}{2}(u_j - u_h)^2 = \frac{5P_{sj}}{2\rho_{sj}} \rightarrow P_{sj} = \frac{1}{5}\rho_{sj}(u_j - u_h)^2$

$= \frac{4}{5}\rho_j(u_j - u_h)^2$

Along similar lines:

$P_{sIGM} = \frac{4}{5}\rho_{IGM}(u_{IGM} - u_h)^2$

$= \frac{4}{5}\rho_{IGM}u_h^2$

d) Contact disc $\rightarrow$ $P_{sj} = P_{sIGM}$

$\rightarrow u_h = \frac{u_j}{1+\sqrt{\eta}}$ with $\eta = \rho_{IGM}/\rho_j$

(16) a) $r_s = \frac{GM}{2C_s^2}$ $\qquad C_s^2 = \frac{k_B T}{m_H} = 1.65 \times 10^{10}\ m^2 s^{-2}$

isothermal

$\rightarrow r_s = 4 \times 10^9 m = 5.8\ R_\odot = 0.027\ AU$

$(7 \times 10^8 m)$

b) $u\frac{du}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2} + Bu\left(\frac{du}{dr}\right)$

$(1-B)u\frac{du}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2}$

$(1-B)u^2\frac{d\ln u}{dr} = -C_s^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}$ $\quad$ with $\frac{d\ln\rho}{dr} = -\frac{2}{r} - \frac{d\ln u}{dr}$

$\rightarrow \left[(1-B)u^2 - C_s^2\right]\frac{d\ln u}{dr} = \frac{2C_s^2}{r}\left[1 - \frac{GM}{2C_s^2 r}\right]$

Critical point: $r_c = \frac{GM}{2C_s^2}$,

$(1-B)u_c^2 - C_s^2$

$\rightarrow u_c = \frac{C_s}{(1-B)^{1/2}}$

(if $B<1$)

(17) a) $\rho_s = \rho_\infty e^{3/2}$

$\dot{M} = 4\pi r_s^2 \rho_s C_s$

$= 4\pi \frac{G^2M^2}{4C_s^4}\rho_\infty e^{3/2} C_s = \pi\frac{G^2M^2}{C_s^3}\rho_\infty e^{3/2}$; $r_s = \frac{GM}{2C_s^2}$

b) $r_s = 6.67 \times 10^{13}\,(M/M_\odot)\,m = 9.5 \times 10^4\,(M/M_\odot)\,R_\odot$

c) $\dot{M} = 4.16 \times 10^{14}\left(\frac{M}{M_\odot}\right)^2 kg\ s^{-1} = 6.56 \times 10^{-9}\left(\frac{M}{M_\odot}\right)^2 M_\odot\ yr^{-1}$

c) $\dot{m} = C m^2$

$$\rightarrow \frac{dm}{dt} = Cm^2 \rightarrow m(t) = (m_0^{-1} - Ct)^{-1}$$

$$= \frac{m_0}{(1 - Cm_0 t)}$$

$$m(t) = 2m_0 \rightarrow Cm_0 t = 1/2$$

$$t_{double} = (Cm_0)^{-1} \frac{1}{2}$$

For $m_0 = 1 M_\odot$ and $\dot{m} = 6.56 \times 10^{-9} M_\odot \, yr^{-1}$

$$\rightarrow t_{double} = 7.6 \times 10^7 \, yr$$

(18) a) Writing Navier-Stokes for cylindrical coordinates

$$u_R: \frac{\partial u_R}{\partial t} + u_r \frac{\partial u_R}{\partial t} + \frac{u_\phi}{r}\frac{\partial u_R}{\partial \phi} + u_z \frac{\partial u_R}{\partial z} - \frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R}$$

$$+ \nu\left(\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 u_R}{\partial \phi^2} + \frac{\partial^2 u_R}{\partial z^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2}\right) - \underset{z \ll R}{\frac{GM}{R^2}}$$

$$u_z: \frac{\partial u_z}{\partial t} + u_R \frac{\partial u_z}{\partial R} + \frac{u_\phi}{R}\frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$+ \nu\left(\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{R}\frac{\partial u_z}{\partial R}\right) - \underset{z \ll R}{\frac{GMz}{R^3}}$$

$$u_z = 0, \quad \frac{\partial}{\partial \phi} = 0, \quad \frac{\partial u_R}{\partial t} = 0$$

$$u_R: \quad u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} + \nu\left(\frac{\partial^2 u_R}{\partial R^2} + \frac{\partial^2 u_R}{\partial z^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2}\right) - \frac{GM}{R^2}$$

$$u_z: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{GMz}{R^3}$$

b) $$H \ll R \rightarrow \left|\frac{\partial p}{\partial z}\right| \simeq \frac{P}{H} \rightarrow \frac{1}{\rho}\frac{P}{H} \simeq \frac{GMH}{R^3}$$

$$\rightarrow \frac{H^2}{R^2} \approx \underbrace{\frac{R}{GM}}_{(R^2\Omega^2)^{-1}} \underbrace{P/\rho}_{c_s^2} \Rightarrow \frac{H}{R} \approx \frac{c_s}{u_\phi}, \text{ or } H\Omega \approx c_s$$

At 1 AU for $M = 1 M_\odot$, $u_\phi \simeq 30$ km/s

$$c_s \sim H\Omega = \frac{H}{R} u_\phi = 0.3 \text{ km/s}$$

$$c_s = \sqrt{\gamma P/\rho} = \sqrt{\frac{\gamma k_B T}{\mu m_H}} \rightarrow T = \frac{\mu m_H}{\gamma k_B} c_s^2 = \frac{\mu}{\gamma} 11 K \simeq 10K$$

c) From the u_R equation, neglecting the viscosity terms:

$$u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} - \frac{GM}{R^2}$$

$$\frac{|\partial P/\partial R \, \frac{1}{\rho}|}{|GM/R^2} \simeq \frac{1/\rho \, P/R}{GM/R^2} = \frac{P/\rho}{GM/R} \simeq \frac{c_s^2}{u_\phi^2} \simeq \frac{H}{R} \ll 1$$

Since $u_R \frac{\partial u_R}{\partial R}$ is also small (at least compared to u_ϕ)

the u_R equation gives $-\frac{u_\phi^2}{R} \simeq -\frac{GM}{R^2}$

$$\rightarrow u_\phi^2 = \frac{GM}{R}, \text{ keplerian}$$

d) A pressure gradient changes the local gravitational force

$$-\frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} - \frac{GM}{R^2}$$

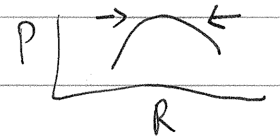
$\Rightarrow$ gravity is weakend if $\partial P/\partial R < 0 \rightarrow u_\phi < u_{\phi,kepler}$

strengthened if $\partial P/\partial R > 0 \rightarrow u_\phi > u_{\phi,kepler}$

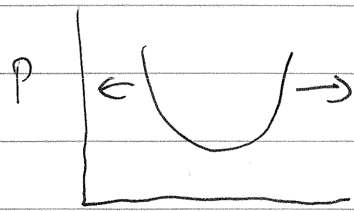
Solid bodies do not experience the pressure force

If $\partial P/\partial R > 0$, the gas has $u_\phi > u_{\phi,kepler}$, so through gas drag the solids will acquire angular momentum and move out.

If the pressure has a local maximum the solids will collect there



If the pressure has a local minimum the solids will move away from it.



e) $L_{disc} = \frac{G M \dot{m}}{2 R_*}$ (12.29)

$L_{disc} = L_\odot \rightarrow \dot{m} = \frac{2 R_\odot L_\odot}{G M_\odot}$

$= 6.5 \times 10^{-8} M_\odot \, yr^{-1}$

$\Rightarrow$ yes, a large part of the luminosity of proto-stars is due to accretion.

19 a) $\rho_j^{n+1} = \rho_j^n - \frac{u \Delta t}{2 \Delta x} \left(\rho_{j+1}^n - \rho_{j-1}^n \right)$

$\rho_j^n = C_k^n \exp(ikj\Delta x)$

$\frac{u \Delta t}{\Delta x} \equiv \lambda$

$$C_k^{n+1} \exp(ikj\Delta x) = C_k^n \exp(ikj\Delta x) - \frac{\lambda}{2}\left[C_k^n \exp(ik(j+1)\Delta x - C_k^n \exp(ik(j-1)\Delta x)\right]$$

$$\rightarrow \frac{C_k^{n+1}}{C_k^n} = 1 - \frac{\lambda}{2}\left(\exp(ik\Delta x) - \exp(-ik\Delta x)\right)$$

$$e^{ix} = \cos x + i\sin x$$

$$\rightarrow e^{ix} - e^{-ix} = 2i\sin x$$

$$\frac{C_k^{n+1}}{C_k^n} = 1 - \frac{\lambda}{2} \cdot 2i\sin(k\Delta x) = 1 - i\lambda\sin(k\Delta x)$$

$$\left|\frac{C_k^{n+1}}{C_k}\right|^2 = \left|1 - \lambda i\sin(k\Delta x)\right| = 1 + \lambda^2\sin^2(k\Delta x) \geq 1$$

$\rightarrow$ always unstable

b) Flux should be of the form ρu and associated with the interface, so

$$F_{j+\frac{1}{2}} = \frac{1}{2}(\rho_j^n + \rho_{j+1}^n)u = \tilde{\rho}_{j+\frac{1}{2}}^n u$$

$$\tilde{\rho}_{j+\frac{1}{2}}^n = \frac{1}{2}(\rho_j^n + \rho_{j+1}^n)$$

c) Use an iterative method

Or, ~~use~~ write the algorithm as a matrix expression and invert the matrix

d)
$$\rho_j^{n+1} = \rho_j^n - u\frac{\Delta t}{2\Delta x}\left(\rho_{j+1}^{n+1} - \rho_{j-1}^{n+1}\right)$$

$$\rho_j^n = C_k^n \exp(ikj\Delta x)$$

$$C_k^{n+1}\exp(ikj\Delta x) = C_k^n \exp(ikj\Delta x) -$$

$$\lambda\left[C_k^{n+1}\exp(ik(j+1)\Delta x) - C_k^{n+1}\exp(ik(j-1)\Delta x)\right]$$

$$\rightarrow \quad C_k^{n+1} = C_k^n - \lambda\left[C_k^{n+1}\left(\exp(ik\Delta x) - \exp(-ik\Delta x)\right)\right]$$

$$\frac{C_k^{n+1}}{C_k^n} = \frac{1}{1+\lambda(\exp(ik\Delta x) - \exp(-ik\Delta x))}$$

$$= \frac{1}{1+\lambda i \sin k\Delta x}$$

$$\left|\frac{C_k^{n+1}}{C_k^n}\right|^2 = \frac{1}{1+\lambda^2\sin^2 k\Delta x} \leq 1$$

$\rightarrow$ always stable

Can be expected, by using information from the future, from the entire domain:

ρ_j^{n+1} depends on ρ_{j-1}^{n+1} which depends on $\rho_{j-2}^{n+1}, \ldots$

(20) a) Isothermal: $P \propto \rho$ → if we know ρ, we know P

b) $1.6 = 2 + \Delta t\,(0.2 - 0.6)$ → $\Delta t = 1$

$2.4 = A + \Delta t\,(0.6 - B)$ → $A = 3$

$2.4 = 2 + \Delta t\,(B - 0.8)$ → $B = 1.2$

$D = 0.6 + \Delta t\,(1.04 - E)$ → $D = -0.54$

$-0.1 = 1.2 + \Delta t\,(E - 3.48)$ → $E = 2.18$

$1.96 = F + \Delta t\,(3.48 - 2.32)$ → $F = 0.8$

c) The states W are

$(2,\ 0.6)$ → $u = 0.3$

$(3,\ 1.2)$ → $u = 0.4$

$(2,\ 0.8)$ → $u = 0.4$

→ $\max(u) = 0.4$

$c_s = 1$

$$\Delta t = \eta_{CFL} \frac{\Delta x}{u + c_s} \rightarrow \eta_{CFL} = (u + c_s)\frac{\Delta t}{\Delta x} = 1.4 > 1$$

↓

Bad

d) Old state: $(2,\ 0.6)$, $(3,\ 1.2)$, $(2,\ 0.8)$

Flux: $(0.2,\ 1.04)$, $(0.6,\ 2.18)$ $(1.2,\ 3.48)$ $(0.8,\ 2.32)$

Notice: $F_{j+1/2}^{\text{dens.}} \Rightarrow W_j^{\text{mom.}}$

Check with momentum flux $\rho u^2 + p = \rho u^2 + \rho c_s^2$
$\rightarrow$ works!

so $F_{j+1/2} = F(W_j)$

This is not a reasonable flux recipe.
The velocities are subsonic ($u < c_s$), so
information should travel left <u>and</u> right.