Astrophysical Gasdynamics Notes:
1. From particles to fluids

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1 Introduction

We want to derive the equations that describe the time evolution of the state of a gas. The first question then becomes: What quantities define the state of a gas?

1. How much there is, so the number density n, or the mass density ρ.
2. How it moves, so the velocity $\mathbf{u}$, or momentum density $\rho\mathbf{u}$ (systemic motion).
3. Internal motions of the gas particles, expressed as pressure p, or temperature T, or internal energy $\rho\mathcal{E}$.
4. Other quantities, such as composition, magnetic field, etc..

The minimum set is the first three, so we need a set of equations describing the time evolution of these quantities.
To arrive at these equations there are different ways. One is to use conservation principles (mass, momentum, energy) and thermodynamics. This is the way it is done in the book. The other way is to use the tools of statistical mechanics, i.e. considering a gas as a collection of particles. Because the latter gives us a better insight in when the equation of gas dynamics apply, and the fact that gas consists of particles is more important in astrophysical contexts, we will follow this path here.

2 Distribution function

Consider a collection of N gas particles of equal mass m. Each particle has a position $\mathbf{x}$ and a velocity $\mathbf{v}$. We can thus put it in a 6-dimensional *phase space* (called μ) of position and velocity, and count the number of particles in the 6-dimensional volume $(\mathbf{x} : \mathbf{x} + \delta\mathbf{x}, \mathbf{v} : \mathbf{v} + \delta\mathbf{v})$. Doing this gives us the *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$:

$$N(\mathbf{x} : \mathbf{x} + \delta\mathbf{x}, \mathbf{v} : \mathbf{v} + \delta\mathbf{v}, t) = f(\mathbf{x}, \mathbf{v}, t)\delta\mathbf{x}\delta\mathbf{v} \tag{1}$$

Now this is something you have seen before. Famous distribution functions are the Bose-Einstein and Fermi-Dirac distribution functions, for bosons and fermions respectively. Since we are not dealing with quantum effects, we can use that other famous distribution function, the Maxwell-Boltzmann distribution function

$$f_{\mathrm{MB}}(\mathbf{v}) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left[-\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2k_{\mathrm{B}}T}\right] \tag{2}$$

which gives the distribution of the particles over velocities in an equilibrium system at temperature T. This distribution function does not depend on position because the system is at equilibrium. The microscopic quantities are the particle mass m, velocity v, mean velocity v_0, and the macroscopic quantities are the temperature T and the number density of particles n. k_{B} is the Boltzmann constant, connecting energy and temperature.

3 Collisionless systems

Equilibrium systems are rather boring, so what we would like is to find a way to describe the time evolution of $f(\mathbf{x}, \mathbf{v}, t)$. What is $\mathrm{d}f/\mathrm{d}t$? Since f is a function of t, $\mathbf{x}$ and $\mathbf{v}$, we can start by writing

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{v}} + \cdots \tag{3}$$

Do we need more terms after that, like $\partial^2\mathbf{v}/\partial t^2$? No, as we know from classical mechanics, if a system can be described with a Hamiltonian $H(\mathbf{x}, \mathbf{v}, t)$ we do not need more equations than

$$\begin{aligned}\dot{\mathbf{v}} &= \mathbf{a} = -\boldsymbol{\nabla} H\\ \dot{\mathbf{x}} &= \mathbf{v} = \boldsymbol{\nabla}_v H\end{aligned} \tag{4}$$

where $\boldsymbol{\nabla}_v$ is the gradient in the velocity coordinate: $(\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$. If we have only outside forces working on the particles, such an H exists, and can be written as $H = u^2/2 + \phi(\mathbf{x})$. However, if the forces on a particle depend on neighbouring particles, for example due to collisions, we cannot find an H which depends only on $\mathbf{x}$ and $\mathbf{v}$.

Let us now consider the case of no collisions, a so-called collisionless system.The evolution of f is then particularly simple, as $\mathrm{d}f/\mathrm{d}t = 0$. This can be shown by realizing that f is a density, and hence must obey the equation of continuity (derived from the principle of mass conservation). As shown in the book (Sect 2.1), this means

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot f\mathbf{u} = 0 \tag{5}$$

where $\mathbf{u}$ is the 'velocity' for the density f. In μ-space this velocity is the six-dimensional vector $(\mathbf{v}, \mathbf{a})$, so the continuity equation becomes

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot(f\mathbf{v}) + \frac{\partial}{\partial \mathbf{v}}\cdot(f\mathbf{a}) = 0 \tag{6}$$

This can be rewritten as

$$\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{v}} + f\left(\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}_v\cdot\mathbf{a}\right) = 0 \tag{7}$$

From the Hamilton relations, Eq. 4 we know that

$$\boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}_v \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}_v H) - \boldsymbol{\nabla}_v \cdot (\boldsymbol{\nabla} H) = 0 \tag{8}$$

So, the evolution of f can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{9}$$

an equation known as the ***Collisionless Boltzmann Equation***. This equation can be used for all kinds of systems consisting of particles, for example low density stellar systems.
If we consider a volume $\delta x \delta v$ in phase space, then it will contain $N = f \delta x \delta v$ particles. Following these particles, they will at a later time be contained in a volume $\delta x' \delta v'$. However, for a collisionless system, f will not have changed in this volume (since $\mathrm{d}f/\mathrm{d}t = 0$), and since particle number N is conserved, this implies $\delta x \delta v = \delta x' \delta v'$, i.e. the shape of the volume can change, but not its total value. This is generally true for a system which can be described by a Hamiltonian.

4 Collisions

For a typical gas, collisions are important, so the equation has to be modified. An important parameters when considering collisions is the mean free path

$$\lambda^{-1} = \sqrt{2} \pi a^2 n \tag{10}$$

for number density n and particle size a. For collisions to matter $\lambda \ll L$, the size of our system. For collisions to be a pertubation (rather than a permanent condition of the collection of particles) we need $\lambda \gg a$. In the latter case we speak of a 'dilute gas'.
For a dilute case, their effects can be seen as a perturbation on the case of no collisions (see Fig. 1), so the Boltzmann equation becomes

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C \tag{11}$$

where C describes the effects of collisions. If we can find a good mathematical description for C we are in business. The description of the effects of collisions on the distribution function was the major achievement of Ludwig Boltzmann.
Remember that we are after the evolution of the gas density, velocity, and energy. But these quantities can be derived from the distribution function f:

$$\rho(\mathbf{x}) = \int m f(\mathbf{x}, \mathbf{v}) d\mathbf{v} \tag{12}$$

$$\mathbf{u}(\mathbf{x}) = \int \mathbf{v} f(\mathbf{x}, \mathbf{v}) d\mathbf{v} / \int f(\mathbf{x}, \mathbf{v}) d\mathbf{v} \tag{13}$$

$$E(\mathbf{x}) = \int \frac{1}{2} m v^2 f(\mathbf{x}, \mathbf{v}) d\mathbf{v} / \int f(\mathbf{x}, \mathbf{v}) d\mathbf{v} \tag{14}$$

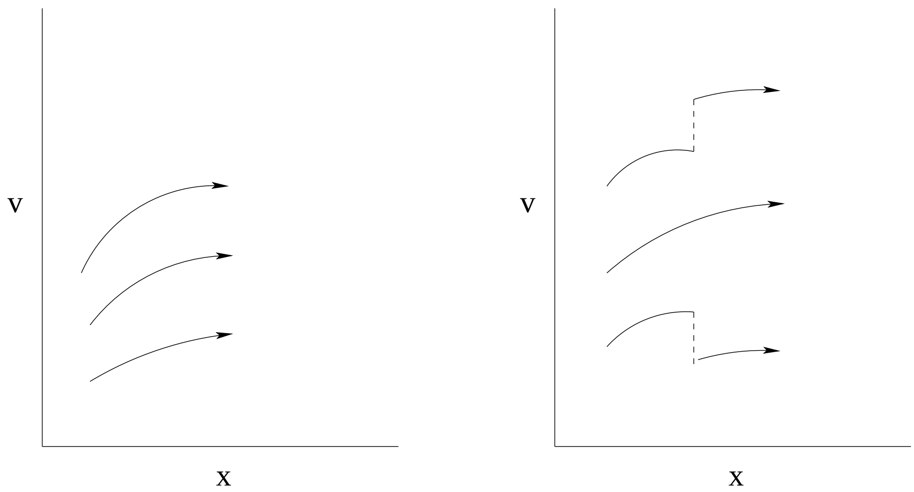

Figure 1: The effect of collisions on the distribution function.

so $\mathrm{d}f/\mathrm{d}t$ will give us $\mathrm{d}\rho/\mathrm{d}t$, $\mathrm{d}\bar{\mathbf{v}}/\mathrm{d}t$, $\mathrm{d}e/\mathrm{d}t$.
We now write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C = C_{\mathrm{in}} - C_{\mathrm{out}} \tag{15}$$

where C_{in} refers to collisions that add to f, and C_{out} to collisions that take away from f (see Fig. 2). We assume that the collisions are binary, and of a short range nature (so the particles are for example not charged).
The number of collisions between two beams of particles with densities n and n_1, and velocities v and v_1 can be written as

$$\delta_{n_c} = nn_1|v - v_1|\sigma\delta\Omega \tag{16}$$

where σ is the cross section for collisions and $\delta\Omega$ the direction of the scattered particles. The densities n and n_1 can be written as $\int f(\mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v}$ and $\int f_1(\mathbf{x}, \mathbf{v}_1)\mathrm{d}\mathbf{v}_1$, so that the term for particles scattered out of $f(\mathbf{x}, \mathbf{v})\delta\mathbf{x}\delta\mathbf{v}$ becomes

$$C_{\mathrm{out}} = \int \mathrm{d}\mathbf{v}_1 \int \mathrm{d}\Omega f f_1 |v - v_1|\sigma \tag{17}$$

.
The particles that collide into $f(\mathbf{x}, \mathbf{v})\delta\mathbf{x}\delta\mathbf{v}$ come from regions of phase space $(\mathbf{x}', \mathbf{v}')$ and $(\mathbf{x}_1', \mathbf{v}_1')$ and thus

$$C_{\mathrm{in}} = \int \mathrm{d}\mathbf{v}_1' \int \mathrm{d}\Omega f' f_1' |v' - v_1'|\sigma \tag{18}$$

.
The next crucial step is to realize that

1. Collisions can be considered to be reversible, i.e. if the collision between particles of velocity $\mathbf{v}$ and $\mathbf{v}_1$ leads to velocities $\mathbf{v}'$ and $\mathbf{v}_1'$, then the reverse is also true.

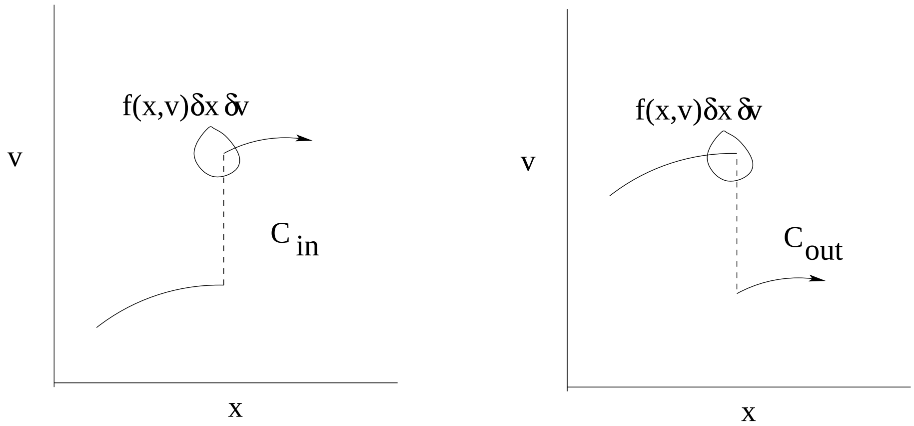

Figure 2: A sketch of the principle behind the $C_{\rm in}$ and $C_{\rm out}$ terms in the derivation of the Boltzmann equation.

2. Conservation of momentum and energy implies that

$$\mathbf{v} + \mathbf{v}_1 = \mathbf{v}' + \mathbf{v}_1' \tag{19}$$

$$\frac{1}{2}|\mathbf{v}|^2 + \frac{1}{2}|\mathbf{v}_1|^2 = \frac{1}{2}|\mathbf{v}'|^2 + \frac{1}{2}|\mathbf{v}_1'|^2 \tag{20}$$

   or equivalently $|\mathbf{v} - \mathbf{v}_1| = |\mathbf{v}' - \mathbf{v}_1'|$

3. A collision between two particles is a Hamiltonian system (in (v, v_1) phase space), so $\delta\mathbf{v}\delta\mathbf{v}_1$=$\delta\mathbf{v}'\delta\mathbf{v}_1'$

This allows us to rewrite $C_{\rm in} - C_{\rm out}$ as one term

$$C = \int \mathrm{d}\mathbf{v}_1 \int \mathrm{d}\Omega |v - v_1| \sigma (f' f_1' - f f_1) \tag{21}$$

where f, f_1, f' and f_1' are the same functions, but evaluated at different velocities $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}'$ and $\mathbf{v}_1'$.
This then leads to the famous ***Boltzmann Equation***

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \int \mathrm{d}\mathbf{v}_1 \int \mathrm{d}\Omega |v - v_1| \sigma (f' f_1' - f f_1) \tag{22}$$

a non-linear integro-differential equation. One should realize that the above derivation cuts some corners, and this equation actually hides quite a bit of interesting physics, and has been the source of many discussions. We will use the Boltzmann equation can be used to derive the equations of gas dynamics.

5 Stationary solution of the Boltzmann equation

It is useful to first consider the solution for the stationary case, without outside forces. For this case $\partial f / \partial t = 0$ and $f(\mathbf{x}, \mathbf{v}, t) = f(\mathbf{v})$, so $\partial f / \partial x = 0$. Without outside forces

$\mathbf{a} = 0$.
All of this implies that the total collisional term C should be zero, or $f' f_1' = f f_1$. Taking the logarithm, this can be rewritten into

$$\log f(\mathbf{v}) + \log f(\mathbf{v}_1) = \log f(\mathbf{v}') + \log f(\mathbf{v}_1') \tag{23}$$

This form reminds one of a conservation law. If a (velocity dependent) quantity $\chi(\mathbf{v})$ is conserved in collisions, then

$$\chi(\mathbf{v}) + \chi(\mathbf{v}_1) = \chi(\mathbf{v}') + \chi(\mathbf{v}_1') \tag{24}$$

So, the stationary solution for the distribution function can be written as a sum of all conserved quantities χ_n

$$\log f(\mathbf{v}) = C_0 + \Sigma_n C_n \chi_n(\mathbf{v}) \tag{25}$$

where the C's are constants. As we saw before, there are only two velocity dependent conserved quantities, namely momentum and energy, so we can write

$$\log f(\mathbf{v}) = C_0 + C_1 |\mathbf{v}|^2 + C_{2x} v_x + C_{2y} v_y + C_{2z} v_z \tag{26}$$

which can be rewritten as $\log f(\mathbf{v}) = \log A - B(v - v_0)^2$ or $f(\mathbf{v}) = A \exp(-B(v - v_0)^2)$. With all the proper normalizations this becomes the Maxwell-Boltzmann distribution function, Eq. 2.
The result is thus that the stationary solution for the Boltzmann equation is the Maxwell-Boltzmann distribution function. This is the reason why f_{MB} is so useful. Any initial condition left to itself will evolve to this distribution. Note however that its derivation relies on binary short range collisions, and the absence of non-conservative forces. When these conditions not hold, f_{MB} will not necessarily be the equilibrium solution.
It is also possible to show that f_{MB} represents the state of maximum entropy, and thus that entropy always increases as a given initial state evolves towards f_{MB}. This is known as Boltzmann's H-theorem, and is of course closely related to the 2nd law of thermodynamics.

6 Macroscopic quantities

As mentioned above, we can derive macroscopic quantities like density and energy from the distribution function f. Let us consider a quantity Q

$$\int \mathrm{d}\mathbf{v} Q f = n \langle Q \rangle \tag{27}$$

The time evolution of Q can be found from the Boltzmann equation.

$$\int \mathrm{d}\mathbf{v} Q \left(\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} \right) = \int \mathrm{d}\mathbf{v} Q C \tag{28}$$

If χ is a quantity which is conserved in binary collisions, one can show that

$$\int \mathrm{d}\mathbf{v} \chi C = 0 \tag{29}$$

This means that we can write the time evolution of such a quantity χ as (using the summation convention)

$$\frac{\partial}{\partial t}\int \mathrm{d}\mathbf{v}\chi f + \frac{\partial}{x_i}\int \mathrm{d}\mathbf{v}\chi v_i f - \int \mathrm{d}\mathbf{v} f v_i \frac{\partial \chi}{\partial x_i} + \tag{30}$$

$$\int \mathrm{d}\mathbf{v}\frac{\partial}{\partial v_i}(\chi f a_i) - \int \mathrm{d}\mathbf{v} f \frac{\partial \chi}{\partial v_i} a_i - \int \mathrm{d}\mathbf{v} f \chi \frac{\partial a_i}{\partial v_i} = 0 \tag{31}$$

The 4th term is zero (through the divergence theorem), so this becomes

$$\frac{\partial}{\partial t} n\langle\chi\rangle + \frac{\partial}{\partial x_i} n\langle v_i \chi\rangle - n\left\langle v_i \frac{\partial \chi}{\partial x_i}\right\rangle - n\left\langle a_i \frac{\partial \chi}{\partial v_i}\right\rangle - n\left\langle \frac{\partial a_i}{\partial v_i}\chi\right\rangle = 0 \tag{32}$$

For an acceleration due to a conservative force, the last term is also zero.

6.1 Mass

For $\chi = m$ the mass of the particles this then gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\mathbf{u} = 0 \tag{33}$$

The ***continuity equation***, describing the evolution of the gas density. Here we have defined the mass density $\rho = nm$ and the mean (or bulk) velocity of the particles as $\mathbf{u} = \langle\mathbf{v}\rangle$.

6.2 Momentum

For $\chi = mv_j$ the momentum in the j direction we get

$$\frac{\partial \rho\langle v_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}(\rho\langle v_i v_j\rangle) - \rho a_j = 0 \tag{34}$$

Now $\langle v_j\rangle = u_j$, but $\langle v_i v_j\rangle \neq v_i v_j$. Let us define the difference between a particle velocity and the mean velocity as $w_i = v_i - u_i$, and construct a tensor P so that

$$P_{ij} = \rho\langle w_i w_j\rangle = \rho(\langle v_i v_j\rangle - u_i u_j) \tag{35}$$

then

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i u_j = -\frac{\partial P_{ij}}{\partial x_i} + \rho a_j \tag{36}$$

The ***momentum or Euler equation***.
Note that the book defines a stress tensor σ_{ij} which is related to P by $\sigma_{ij} = \rho\langle v_i v_j\rangle = P_{ij} + \rho u_i u_j$.

6.3 Energy

For $\chi = \frac{1}{2}mv^2$ the kinetic energy of the particles, it makes sense divide this into the energy connected with the mean velocity $\frac{1}{2}mu^2$ and the remainder $m\mathbf{u}\cdot\mathbf{w} + \frac{1}{2}mw^2$. As $\langle \mathbf{u}\cdot\mathbf{w}\rangle = \mathbf{u}\cdot\langle\mathbf{w}\rangle = 0$ since $\langle\mathbf{w}\rangle = 0$, the equation becomes

$$\frac{\partial}{\partial t}\rho(u^2 + \langle w^2\rangle) + \frac{\partial}{\partial x_i}\rho\langle(u_i + w_i)|\mathbf{u}+\mathbf{w}|^2\rangle = \rho\mathbf{u}\cdot\mathbf{a} \tag{37}$$

The $\langle(u_i + w_i)|\mathbf{u}+\mathbf{w}|^2\rangle$ term can be written out as follows

$$\langle(u_i + w_i)(u_i + w_i)^2\rangle = u^2 u_i + 2\mathbf{u}\cdot\langle\mathbf{w}w_i\rangle + u_i\langle w^2\rangle + \langle w_i w^2\rangle \tag{38}$$

We define the internal energy of the gas as $\rho\mathcal{E} = \frac{1}{2}\rho w^2$, and the conduction heat flux as $\mathbf{q} = \rho\langle\frac{1}{2}w^2\mathbf{w}\rangle$, then

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial x_i}u_i\left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}u_j P_{ij} = \rho\mathbf{u}\cdot\mathbf{a} \tag{39}$$

The ***energy equation***. We will call $\frac{1}{2}\rho u^2 + \rho\mathcal{E}$ the total energy E.

6.4 Closure relation

These three equations are general but not a closed set since we have 5 equations, but 13 quantities: ρ, $\mathbf{u}$, $\mathcal{E}$, and P[1].
What can we do with the excess unknowns? They seem to be related to the miscroscopic behaviour of the fluid, and so we need the distribution function to say something about them. For an equilibrium case, this is f_{MB}, but this would be very boring. So, let us assume *local* Maxwell-Boltzmann. In this case

$$P_{ij} = \rho\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\int \mathrm{d}\mathbf{w}w_i w_j \exp\left(-\frac{mw^2}{2k_{\mathrm{B}}T}\right) = nk_{\mathrm{B}}T\delta_{ij} \tag{40}$$

a diagonal matrix, and $\rho\mathcal{E} = \frac{3}{2}nk_{\mathrm{B}}T$. Only the diagonal terms survive because the off-diagonal terms are asymmetric around zero and thus give no contribution under the integral. Similarly, one can show that the heat conduction flux $\mathbf{q}$ is equal to zero (since it involves a symmetric integral over the asymmetric function $w_i w^2$).
Now as we know from thermodynamics, the term $nk_{\mathrm{B}}T$ is the gas pressure p. We will therefore use this instead of the temperature.
The set of equations thus becomes

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\mathbf{u} = 0 \tag{41}$$

$$\frac{\partial\rho u_i}{\partial t} + \boldsymbol{\nabla}\cdot\rho u_i\mathbf{u} = -\nabla_i(p) + \rho a_i \tag{42}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(E+p)\mathbf{u} = \rho\mathbf{u}\cdot\mathbf{a} \tag{43}$$

[1]Note that $\rho\mathcal{E} = \frac{1}{2}\rho\langle w^2\rangle$ and that $P_{ij} = \rho\langle w_i w_j\rangle$, so that $\Sigma_i P_{ii} = 2\rho\mathcal{E}$, thus we have 13 quantities, not 14.

known as the set of **Euler equations for an inviscid fluid**.
Now these equations are not the full fluid equations, since they assume that $f = f_{\rm MB}$ everywhere (although not necessarily the *same* $f_{\rm MB}$). Obviously in real fluids there will be deviations from $f_{\rm MB}$, and these give rise to so-called 'transport phenomena' such as viscosity and thermal conduction. However, in astrophysics these transport phenomena are often unimportant and many astrophysical systems can be described with the set of Euler equations. We will later return to consider the terms introduced by deviations from a local Maxwell-Boltzmann distribution.
It is important to realize that we have removed the information about the motions of the individual particles through the introduction of the tensor P, which occurs in the Euler equations through the pressure terms. So, you could say that the appearance of pressure terms is our punishment for not wanting to deal with the motions of inidividual particles.