Astrophysical Gasdynamics Notes:
3. Transport phenomena

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1 Deviations from Maxwell-Boltzmann

In deriving the Euler equations we have assumed that the distribution function can locally be described as $f_{\rm MB}$. However, this is of course a simplification. Even if we have two regions where this locally true, once exposed to each other, collisions between particles that belong to different $f_{\rm MB}$ distributions will push the evolution of the system towards a new equilibrium $f_{\rm MB}$.
The interaction of particles belonging to different distribution functions gives rise to so-called transport phenomena. As the Maxwell-Boltzmann distribution is characterized by v and T we can expect the new terms associated with these transport phenomena to have to do with differences in u and T.
We return to the Boltzmann equation, describing the evolution of the distribution function,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C \tag{1}$$

and instead of $f = f_{\rm MB}$ take $f = f^{(0)} + g$, where $f^{(0)} = f_{\rm MB}$ and g is a small deviation. Only keeping the first order terms, the collision term now becomes

$$\int \mathrm{d}\mathbf{v}_1 \int \mathrm{d}\Omega |v - v_1| \sigma \left(f^{(0)\prime} g_1' + f_1^{(0)\prime} g' - f^{(0)} g_1 + f_1^{(0)} g \right) \tag{2}$$

One can argue that an order of magnitude estimate for the collision term is

$$v_{\rm rel} \sigma n g \approx \frac{v_{\rm rel}}{\lambda} g = \frac{g}{\tau} \tag{3}$$

where τ is the typical time between collisions. This argument suggest the BGK approximation (Bhatnagar, Gross & Krook):

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{g}{\tau} = -\frac{f - f^{(0)}}{\tau} \tag{4}$$

which means that f relaxes exponentially to the equilibrium value $f^{(0)}$ in a time τ.

2 Chapman-Enskog expansion

To proceed further we use the so-called Chapman-Enskog expansion, in which f is approximated as a series of deviations from the Maxwell-Boltzmann distribution

$$f = f^{(0)} + \alpha f^{(1)} + \alpha^2 f^{(2)} + \cdots \tag{5}$$

where α is a measure of the role of collisions: $\alpha = \lambda/L$ (L being the size of the domain). The first step is to take $f^{(0)}$ as the first approximation and solve for $f^{(1)}$ in the collision term of the Boltzmann equation:

$$-\frac{g}{\tau} = \frac{\partial f^{(0)}}{\partial t} + \mathbf{v} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} \tag{6}$$

Since $f^{(0)}$ is a Maxwell-Boltzmann distribution, it only depends on n, $\mathbf{u}$ and T, which all only depend on time t and position $\mathbf{x}$. So,

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{u}} \tag{7}$$

and similar expressions for $\partial f^{(0)}/\partial x_i$.
Putting f_{MB} into Eq. (7) and the equivalent expressions for the spatial derivative, and then substituting these into Eq. (6), leads to an expression for g

$$g = -\tau \left[\frac{1}{T}\frac{\partial T}{x_i} w_i \left(\frac{m}{2k_{\mathrm{B}}T} w^2 - \frac{5}{2} \right) + \frac{m}{k_{\mathrm{B}}T} \Lambda_{ij} \left(w_i w_j - \frac{1}{3}\delta_{ij} w^2 \right) \right] f^{(0)} \tag{8}$$

where we defined $\mathbf{w} = \mathbf{v} - \mathbf{u}$ (the random velocities of the particles, as before), and

$$\Lambda_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{9}$$

the shear in the macroscopic velocity field.

3 Macroscopic quantities

We can now derive macroscopic quantities from this first order approximation for the distribution function $f = f^{(0)} + g$. Many terms remain the same as before (when f was $f^{(0)}$) because integrals over terms that are odd in w_i give zero. This is to be expected since the new terms should come from *differences* in the flow.

3.1 Heat conduction

We defined the heat flux as

$$\mathbf{q} = \frac{1}{2}\rho \langle w^2 \mathbf{w} \rangle \tag{10}$$

which was zero for $f = f^{(0)}$. For g the $\partial T/\partial x_i w_i$ terms result in an integral over w^2, an even term in w_i. Evaluating

$$\mathbf{q} = \frac{\rho}{2n}\int \mathrm{d}\mathbf{w}\mathbf{w}w^2 g \tag{11}$$

gives

$$\mathbf{q} = -K\boldsymbol{\nabla}T \tag{12}$$

$$K = \frac{\tau m}{6T}\int \mathrm{d}\mathbf{w}w^4\left(\frac{m}{2k_{\mathrm{B}}T}w^2 - \frac{5}{2}\right)f^{(0)} = \frac{5}{2}\tau n\frac{k_{\mathrm{B}}^2 T}{m} \tag{13}$$

This is the transport of internal energy due to the existence of a temperature gradient: ***Heat conduction*** or thermal conduction. The book treated this in Sect. 4.4.2 as a process for energy transport. Here we see that it is in fact due to transport phenomena due to deviations from Maxwell-Boltzmann distribution.

3.2 Viscosity

Before we defined a tensor $P_{ij} = \rho\langle w_i w_j\rangle$. For $f = f^{(0)} = f_{\mathrm{MB}}$ we showed this tensor to be diagonal, and the diagonal elements to be associated with the gas pressure: $P_{ij} = p\delta_{ij}$. For $f = f^{(0)} + g$, the Λ_{ij} term in g adds non-zero off-diagonal terms, $P_{ij} = p\delta_{ij} + \sigma'_{ij}$, with

$$\sigma'_{ij} = m\int \mathrm{d}\mathbf{w}w_i w_j g \tag{14}$$

$$= -\frac{\tau m^2}{k_{\mathrm{B}}T}\Lambda_{kl}\int \mathrm{d}\mathbf{w}w_i w_j\left(w_k w_l - \frac{1}{3}\delta_{kl}w^2\right)f^{(0)} \tag{15}$$

This $\underline{\sigma}$ tensor is traceless (zeros on the diagonal) $\sigma'_{ii} = 0$ and symmetric $\sigma'_{ij} = \sigma'_{ji}$, and is proportional to Λ_{kl}. However, $\Lambda_{kk} = \boldsymbol{\nabla}\cdot\mathbf{u}$, and so not necessarily zero. This suggests that we can write

$$\sigma'_{ij} = -2\eta\left(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u}\right) \tag{16}$$

where the second term between the brackets makes sure that the total expression is traceless.
The coefficient η must follow from the evaluation of the integral (Eq. 15), for example for σ'_{12}

$$\sigma'_{12} = -\frac{\tau m^2}{k_{\mathrm{B}}T}\Lambda_{kl}\int \mathrm{d}\mathbf{w}w_1 w_2\left(w_k w_l - \frac{1}{3}\delta_{kl}w^2\right)f^{(0)} \tag{17}$$

$$= -2\frac{\tau m^2}{k_{\mathrm{B}}T}\Lambda_{12}\int \mathrm{d}\mathbf{w}w_1^2 w_2^2 f^{(0)} \tag{18}$$

since the integral is only non-zero when k and l are a combination of 1 and 2. From this we find

$$\eta = \frac{\tau m^2}{k_{\mathrm{B}}T}\int \mathrm{d}\mathbf{w}w_1^2 w_2^2 f^{(0)} = \tau n k_{\mathrm{B}}T \tag{19}$$

The tensor σ'_{ij} has to do with non-diagonal terms of Λ_{ij}, so velocity variations perpendicular to the velocity direction, an effect known as *shear*. The property of fluids associated with this is known as ***viscosity***, and σ'_{ij} is known as the *viscous stress tensor*, and η is the viscosity coefficient. Interestingly, the above derivation shows that since $\tau = \lambda/\langle v \rangle$,

$$\eta = \frac{1}{4a^2}\sqrt{\frac{mk_{\mathrm{B}}T}{\pi}} \tag{20}$$

independent of the density of the gas! This seems counter-intuitive (many would say that "denser fluids are more viscous") but is in fact confirmed by experiments. The reason is that although a denser gas has more particles to transport physical quantities, the mean free path of these particles is shorter, and they are thus less efficient tranporters. One also sees that $\eta \propto \sqrt{T}$, which is understandable since with higher particle velocity, physical quantities should be transported further. Note that this temperature dependence only holds for gases. For liquids, viscosity goes down with temperature.

4 Navier-Stokes equations

These new effects, conduction and viscosity now have to be added to the fluid equations. The continuity equation does not change. The momentum equation now has a more complicated tensor P_{ij}, namely $P_{ij} = -p\delta_{ij} + \sigma'_{ij}$ and thus can be written as

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underline{\mathsf{P}} + \rho\mathbf{a} = -\boldsymbol{\nabla}p + \eta\left[\boldsymbol{\nabla}^2 u + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right] + \rho\mathbf{a} \tag{21}$$

Note that for some special fluids, there is also a so-called *bulk viscosity* ζ which is associated with a $\boldsymbol{\nabla}\cdot\mathbf{u}$ term:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla}p + \eta\left[\boldsymbol{\nabla}^2 u + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right] + \zeta\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \rho\mathbf{a} \tag{22}$$

This bulk viscosity is associated with diagonal elements for the viscous stress tensor $\underline{\sigma}$, which do not follow from the ideal, monatomic gas-type approach we used to derive $\underline{\sigma}$. The bulk viscosity is associated with internal degrees of freedom of the particles in a non-ideal gas, which can be excited or de-excited through volume changes. It is generally unimportant in astrophysical applications.
The energy equation becomes

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(E + \underline{\mathsf{P}})\mathbf{u} - \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T) = \rho\mathbf{u}\cdot\mathbf{a} \tag{23}$$

which by taking the $p\delta_{ij}$ part out of P_{ij} can be written as

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(E + p)\mathbf{u} + \boldsymbol{\nabla}\cdot(\underline{\sigma}'\mathbf{u}) - \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T) = \rho\mathbf{u}\cdot\mathbf{a} \tag{24}$$

The $\boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T)$ term only acts on the internal energy $\rho\mathcal{E}$, but the complicated $\boldsymbol{\nabla}\cdot(\underline{\sigma}'\mathbf{u})$ term (remember that $\underline{\sigma}'$ is a tensor here) has contributions both to the kinetic and the internal energy. Manipulation of the equations shows that the *viscous heating term* is given by

$$2\eta\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\boldsymbol{\nabla}\cdot u)^2\right] \tag{25}$$

and is an energy sink for kinetic energy, and an energy source for the internal energy. This shows that viscosity is an irreversible, dissipative process through which kinematic energy is turned into internal energy. It is often a small term in the equations.

The new set of equations for ρ, $\mathbf{u}$ and E is called the ***Navier-Stokes Equations***. They are similar to the Euler equations but contain extra terms of *higher* spatial derivatives of the velocity and the temperature. This makes them harder to solve, but also introduces the necessity for more boundary conditions.