Astrophysical Gasdynamics Notes:
4. Turbulence

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Instabilities and perturbations may ultimately lead to a state of random density, velocity and pressure variations, known as turbulence. Turbulence is important in astrophysics and earth-based applications, but turns out to be extremely difficult to describe. In fact, no general theory for turbulence exists.

Since we are talking about random variations, a theory of turbulence has to be statistical. In some sense one can argue that one has to construct another layer of statistics on top of the particle picture

$$\mathbf{v} = \mathbf{w} + \mathbf{u} = \mathbf{w} + \overline{\mathbf{u}} + \mathbf{u}' \tag{1}$$

Since by construction $\overline{\mathbf{u}'} = 0$, analysis of turbulence is about higher order terms $\overline{\mathbf{u}'\mathbf{u}'}$ (just as it was about w^2 in the statistical treatment of particle velocities. As the w^2 term led to pressure or internal energy of the gas, also turbulence studies focus on the turbulent energy.

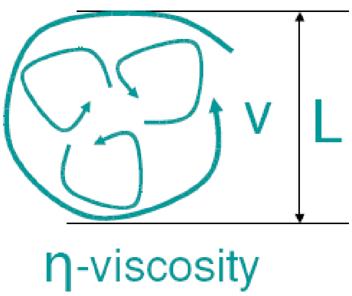

Figure 1: Hierarchy of turbulent eddies.

Unlike the random particle velocities, turbulence has a scale, meaning that turbulent flows have structure. The image often used is that of turbulent 'eddies': a hierarchy of bigger eddies containing smaller eddies, that again consist of even smaller eddies, up to the scales where dissipation sets in. This image led to the *Kolmogorov* picture of the distribution of turbulent energy over length scales

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{2}$$

where ϵ in the energy input rate, and k is the Fourier wavelength, $2\pi/l$, l the length scale. This relation was derived by Kolmogorov from heuristic and dimensional arguments, but seems to give a fairly accurate description of the energy spectrum of turbulence.

As we defined before the Reynolds number $\mathcal{R}$ for a system of size L, velocity V and kinematic viscosity ν is given by

$$\mathcal{R} = \frac{LV}{\nu} \tag{3}$$

If a region is turbulent, the $\mathcal{R}$ is large, or in other words, viscosity will suppress turbulence. The biggest eddies in the system will have $LV/\nu \gg 1$, and smaller eddies will have smaller values of $\mathcal{R}$, until at some scale l_d, the condition $l_d u_d/\nu \sim 1$ is reached, and the turbulent energy is dissipated.

If the energy is fed in at the largest scales L and V at a rate per unit mass ϵ, then for a steady flow of energies from large to small scales

$$\epsilon \sim u^3/l \tag{4}$$

(from dimensional arguments) implying that $u \sim (\epsilon l)^{1/3}$. At the scale of the system L, V this should also hold, defining the input energy rate per mass unit as $\epsilon \sim V^3/L$. At the dissipation scale l_d we have $l_d u_d \sim \nu$, and so $l_d \sim \nu^{3/4}\epsilon^{-1/4}$ and $u_d \sim (\nu\epsilon)^{1/4}$. This then implies that

$$\frac{L}{l_d} \sim \mathcal{R}^{3/4} \tag{5}$$

$$\frac{V}{v_d} \sim \mathcal{R}^{1/4} \tag{6}$$

So given the Reynolds number of the system the dissipation scale and velocity can be found.

To get the energy spectrum $E(k)$, one should realize that that $k \sim l^{-1}$. At scale k the energy is given by

$$E(k)\mathrm{d}k \sim E(k)k \sim u^2 \sim (\epsilon l)^{2/3} \sim (\epsilon/k)^{2/3} \tag{7}$$

which then gives Eq. 2. One can also derive this from dimensional analysis assuming that $E(k)$ only depends on k and ϵ. Obviously, the spectrum will be cut off at small k because of the size of the system (L) and at high k because the dissipation scale is reached (k_d). Experiments for many different systems show this relation to be valid and quite universal.

Since $\mathcal{R}$ is large for most astrophysical systems, turbulence occurs frequently. Examples are

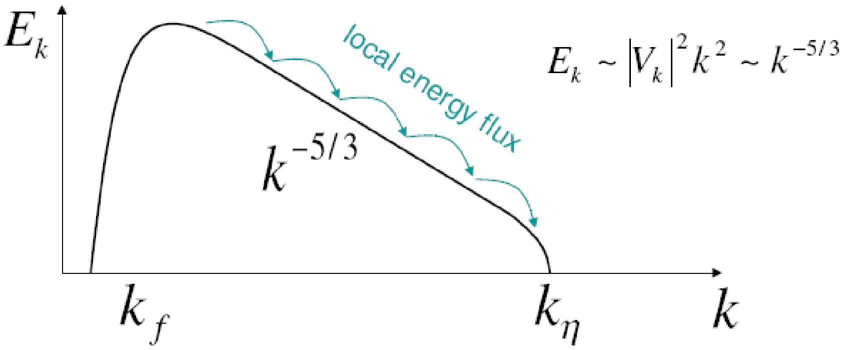

Figure 2: The Kolmogorov energy spectrum for turbulence. The k's corresponding to the size of the system and the dissipation scale are called k_f and k_ν respectively.

- turbulent convection in stars
- turbulence in molecular clouds
- turbulent boundary layers around jets
- atmospheric turbulence causing astronomical seeing
- ...