Stockholms universitet
Institutionen för astronomi
Astrophysical Gasdynamics (AS 7002)
Exam

January 22, 2007, 9am - 2pm

This exam consists of 5 problems with in total 20 subproblems. Each subproblems (marked with a boldface letter) is worth 5 points. For a pass ('G') 60 points are required, for a pass with distinction ('VG') 80 points are required. Answers can be given in either English or Swedish.

Good luck!

1

A) To study the aerodynamic properties of a jet plane, engineers make a scale model which is 100 times smaller than the real plane, and place it in a wind tunnel. How large does the air speed in the windtunnel need to be to study the behaviour of the real plane at 900 km/h? What problem will the engineers run into when they try this experiment? Can you suggest any solutions?

B) Engineers study the structure of the turbulent wake of a car in a wind tunnel. The car is about 5 m long and the wind speed in the tunnel is 30 km/h. What is the Reynolds number of the flow? Argue that this value points to the formation of turbulence. How good a spatial resolution do the engineers approximately need to resolve all of the turbulence? (Use the viscosity data for air from the appendix of the book).

C) Explain the physical reason why both the thermal conduction constant K and the viscosity μ increase with the collision time τ between particles in the gas.

2

A shock whose strength is characterized by Mach number 4 travels into a medium of (number) density $n_1 = 1$ cm^{-3}, temperature $T_1 = 100$ K and velocity $v_1 = 0$ km $^{-1}$.

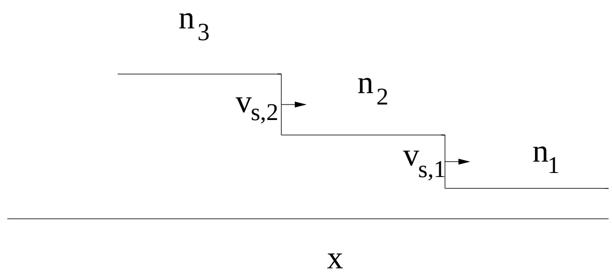

Figure 1: Two shocks travelling with speeds $v_{s,1}$ and $v_{s,2}$ through a medium with original density n_1.

Assume the medium to consist of neutral hydrogen and take the adiabatic index to be $\gamma = 5/3$.

A) What is the shock velocity of this shock, and what are the post-shock density, velocity and temperature?

This shock is followed by a second shock, this one characterized by a Mach number of 2 (see Fig. 1).

B) Show that the shock velocity of this second shock is higher than that of the first shock, and also here derive the post-shock density, velocity and temperature.

It can be shown that a second shock is *always* faster than the first one, or in other words, the second shock will always catch up with the first one. Here you are asked to prove this for two *strong* shocks for which both Mach numbers can be taken to be much larger than the other terms in the shock jump conditions.

C) What are the shock jump conditions for a strong shock?

D) Show that for two subsequent strong shocks, the second one always moves faster than the first one.

3

A rocket engine works using the principle of the de Laval nozzle.

A) Explain how a de Laval nozzle works (use both words and equations).

The standard de Laval nozzle is oriented horizontally, so there is no gravitational force working in the direction of the flow. For a rocket one typically uses a vertical de Laval nozzle.

B) Show that if the gravitational acceleration is $-g$, the equation for the de Laval nozzle can be written as

$$(1 - \mathcal{M}^2)\frac{1}{v}\frac{dv}{dx} = -\frac{1}{A}\frac{dA}{dx} + \frac{g}{c_s^2} \qquad (1)$$

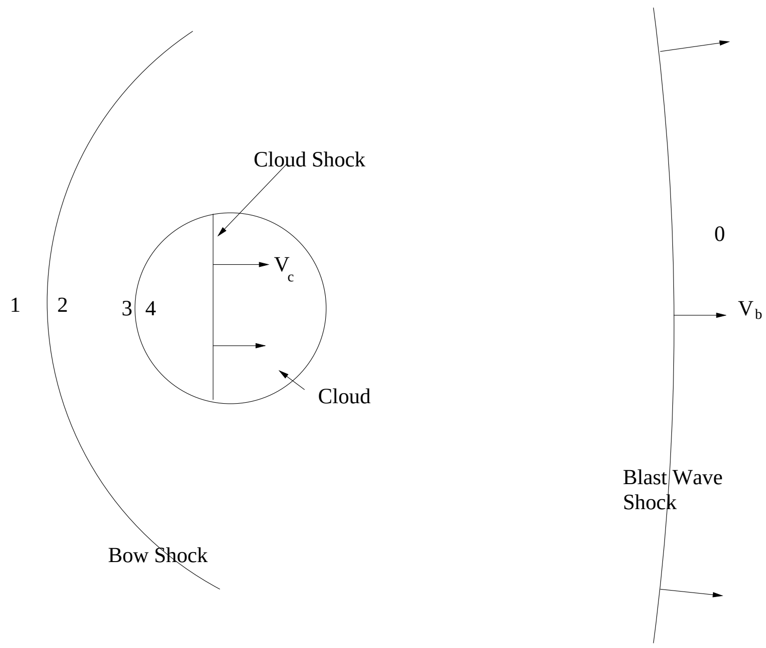

Figure 2: A dense cloud caught in a supernova blast wave of velocity v_b. 0 is the unperturbed intercloud medium, 1 is the shocked intercloud medium. The interaction of the shocked intercloud medium with the cloud leads to a shock travelling into the cloud with velocity v_c.

where A is the cross section of the nozzle.

C) Show that this means that the sonic point for this engine will lie above the narrowest part of the nozzle.

4

A blast wave from a supernova is travelling through the interstellar medium (ISM) with a velocity v_b and encounters a dense cloud of density ρ_c. Before the passage of the shock wave there are no motions in the gas. Figure 2 illustrates the case.
First we consider the change in the intercloud medium caused by the passage of the blast wave.

A) Show that if the ISM pressure p_0 is negligible, the pressure of the intercloud material shocked by the blastwave (called '1') can be written as

$$p_1 = \frac{2}{\gamma + 1} \rho_0 v_b^2 \,, \tag{2}$$

where ρ_0 is the density of the unshocked ISM. (Hint: use a combination of shock jump conditions and flux relations across a shock.)

B) Show that under the same assumption the post-shock Mach number of the intercloud ISM (measured in the frame at rest with the unshocked ISM) is given by

$$\mathcal{M}_1^2 = \frac{2}{\gamma(\gamma - 1)} \tag{3}$$

and show that this means that the shocked intercloud ISM is (initially) is moving supersonically with respect to the cloud.
(The word initially is there in the previous sentence because for a blast wave following for example the Sedov-Taylor solution for a blast wave, the pressure will drop when one moves away from the shock front.)

Because medium 1 has a supersonic velocity with respect to the cloud, a bow shock will form around the cloud. In this bow shock the (already shocked) intercloud ISM is shocked again ('3'). At the same time a shock travels into the cloud with velocity v_c. The shocked cloud material is indicated with '4'.
We will now try to find a value for v_c expressed in terms of the ISM density ρ_0, the cloud density ρ_c, and the blast wave velocity v_b.
The first step is to realize that at the interface 3-4 the pressures and velocities are the same on either side of the interface.

C) Why are these pressures and velocities the same?

Assuming the shock travelling into the cloud to be strong, we can use the same arguments as under **A)** to show that

$$p_4 = \frac{2}{\gamma + 1}\rho_c v_c^2 \, , \tag{4}$$

To derive the relation between v_c, v_b, ρ_c and ρ_0, we need to connect the pressure p_1 to the pressure p_4. This can be done using a combination of shock jump conditions and Bernouilli's principle in the frame of reference in which the velocity at the stagnation point 3-4 is zero.

D) Show that in this frame of reference the Mach number of the shocked intercloud medium ('1') be written as

$$\mathcal{M}_1^{*2} = \mathcal{M}_1^2(1 - x)^2 \, , \tag{5}$$

where $\mathcal{M}_1^2$ is the value from Eq. (3) and $x = v_c/v_b$. Argue that the value of x has to lie between 0 and 1.

The result of connecting point 1 to 4 is that

$$\frac{p_4}{p_1} = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/(\gamma-1)} \mathcal{M}_1^{*2} \left(\gamma - \frac{\gamma - 1}{2\mathcal{M}_1^{*2}}\right)^{-1/(\gamma-1)} . \tag{6}$$

E) Explain in detail how one can connect point 1 to point 4 in this way, and why the ratio can be expressed solely in terms of $\mathcal{M}_1^2$ (you do not need to do the full derivation, just show how you would proceed).

F) With the above results argue that

$$v_c = F \frac{v_b}{\sqrt{\rho_c/\rho_0}} \,. \tag{7}$$

What range of values can F take?

G) Do you expect any Kelvin-Helmholtz instabilities to occur in this problem? And Rayleigh-Taylor? Please motivate.

5

Consider the one-dimensional linear advection equation

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = 0 \,, \tag{8}$$

where v is constant in time and space. An explicit numerical scheme for solving this equation can be written in general form as

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x} (F_{j+1/2} - F_{j-1/2}) \,, \tag{9}$$

where the flux $F_{j+1/2}$ is calculated from some combination of ρ_j^n and ρ_{j+1}^n. The index j indicates the spatial discretization, and n the temporal. Δt is the time step, and Δx the mesh spacing. To be stable such methods always have to fulfill the CFL condition for the time step: $\Delta t < \Delta x / v$.

A) Explain the physical reason for the CFL condition.

The BTCS (Backward Time, Central Space) algorithm for the one-dimensional linear advection equation can be written as

$$\rho_i^{n+1} = \rho_i^n - v \frac{\Delta t}{2 \Delta x} (\rho_{i+1}^{n+1} - \rho_{i-1}^{n+1}) \,. \tag{10}$$

A method like this in which the fluxes depend on the solution for the next time t_{n+1} is called *implicit*.

B) Describe how the use of an implicit method differs from that of an explicit method.

C) Use the technique of von Neumann stability analysis to show that this method is unconditionally stable, i.e. there is not even a CFL condition.