Astrophysical Gasdynamics: Exercises I

November 14, 2007

1 Collisional fluid

Consider a gas of neutral hydrogen atoms of number density n. Calculate the size of the region over which it can be considered to be a collisional fluid for densities of 10^{-4} cm^{-3}, 1 cm^{-3} and 10^4 cm^{-3}. Take the size of the H atoms to be 0.12nm. Express these sizes in parsecs and AUs. On the basis of this argument, can one always use the collisional fluid approximation for astrophysical gases?

2 Static solution of the Boltzmann equation

Go through all the steps that show that the static solution to the Boltzmann equation is of the form $f(\mathbf{v}) = A\exp(-B(v - v_0)^2)$, i.e. it is the Maxwell-Boltzmann distribution function.

3 Lagrangian form

From the Eulerian form derive the Lagrangian form for the continuity, momentum and total energy equation. Also derive the Eulerian and Langrangian forms of the differential equation for the velocity (the equations for $\partial u/\partial t$ and $\mathrm{D}/\mathrm{D}t$).

4 Energy equations

a) The equation for the total energy can be split into a part for the kinetic energy $\frac{1}{2}\rho u^2$ and the internal energy $\rho\mathcal{E}$ Derive these two equations (for inviscid flow) using the total energy equation and the momentum equation (first step: multiply the momentum equation with u for this).
b) The entropy per unit mass is defined as $s = C_V \log(p/\rho^\gamma)+$constant (with C_V the specific heat at constant volume and γ the adiabatic index). Show that the equation for the internal energy derived under a) implies entropy conservation. When is it allowed to solve the internal energy equation instead of the total energy equation?
c) Show that for an isothermal gas you can use $\gamma = 1$. What energy equation can be used for an isothermal gas?

5 Advection

a) Consider the one-dimensional continuity equation for a flow of constant velocity u:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_0}{\partial x} = 0\,. \tag{1}$$

We define an initial condition $\rho(x, t = 0) = \rho_0(x)$. Show that the solution for $\rho(x, t)$ is that $\rho_0(x)$ is 'advected' with velocity u_0, or in mathematical terms $\rho(x, t) = \rho_0(x - u_0 t)$.
If $\rho_0(x)$ is different for every x (e.g., if $\rho_0(x) = x$), sketch the lines of constant density in the x-t plane. These lines are the particle or fluid element paths (in the x-t plane), and in the Lagrangian form, one follows the flow quantities (such as the density) along these paths. Show that this means that in the Lagrangian form, $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, or in other words, ρ does not change along the particle paths.
b) Now consider a slightly more complicated case where $u = x$, which makes the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho x}{\partial x} = 0\,, \tag{2}$$

with the initial condition $\rho(x, t = 0) = \rho_0(x)$.
Argue that in this case, ρ does change along the particle paths. Also argue that we can solve this equation for $x > 0$ without considering the $x < 0$ domain.
We solve this problem in two steps. First we find the evolution of ρ along the particle paths, then we will find the particle paths.
Show from the Lagrangian form of the continuity equation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial u}{\partial x} = 0\,. \tag{3}$$

that along a particle path $\rho(t) = \rho_0 \exp(-t)$.
Now show that the particle paths are of the form $t = \log x + \text{constant}$, and that consequently the full solution is

$$\rho(x, t) = \rho_0(xe^{-t}) \exp(-t) \tag{4}$$

Sketch some particle paths in the x-t plane and qualitatively explain the result found for $\rho(x, t)$ (think in terms of mass conservation).

6 Cosmic rays

Measurements of the high energy particles known as cosmic rays entering the Earth's atmosphere show that their energy distribution follows a power law. Can the dynamics of cosmic rays be described with the fluid equations?

7 Isothermal atmosphere

Clarke & Carswell derived the density distribution in the Earth atmosphere by neglecting the curvature of the Earth's surface, and found the exponential atmosphere (Sect. 5.3). In this problem we will derive the full solution for an isothermal around a spherical body.

Consider an isothermal atmosphere around a spherical body of mass M and radius r_0. Neglect the mass of the atmosphere itself. Write down the gravitational potential as a function of radius r. Use the condition of hydrostatic equilibrium in spherical coordinates to find the density as a function of r. Use $\rho(r_0) = \rho_0$ as the boundary condition.

Show that for $r = r_0 + \Delta r$ and $\Delta r \ll r_0$, one retrieves the solution for a planar atmosphere that was derived in Clarke & Carswell (Eq. 5.24).