Astrophysical Gasdynamics: Exercises III

December 5, 2007

12 Self-similar problems

Show using dimensional arguments that if the shock position for a blast wave is only a function of time t, blast energy E and density ρ_1, it can *only* depend on these quantities as

$$r \propto \left(\frac{Et^2}{\rho_1}\right)^{1/5} \tag{1}$$

Use similar arguments to show that if the energy input is continuous, and can be given by a rate $\dot{E}$, the shock position will instead vary as

$$r \propto \left(\frac{\dot{E}t^3}{\rho_1}\right)^{1/5} \tag{2}$$

An example of this is a stellar wind running into an environment of density ρ_1. Argue that for a supersonic wind with velocity $u_{\rm w}$ and mass loss rate $\dot{M}$, $\dot{E}$ is given by $\frac{1}{2}\dot{M}u_{\rm w}^2$.

13 Bernoulli's equation

a) Show that for an adiabatic gas (with adiabatic index γ), Bernoulli's equation can also be obtained from the total energy equation.
b) A container of height h is filled with a fluid, and has a little hole near the bottom through which fluid escapes. Use Bernoulli's equation to show that the outflow velocity through the hole is $\sqrt{2gh}$ (and does not depend on the size of the hole), where g is the gravitational acceleration. This is known as Torricelli's Theorem (see Fig. 1).

14 De Laval Nozzle

In the lecture we looked at the de Laval nozzle which is a device for producing supersonic flows. For it to work the flow has to make the (smooth) transition from subsonic to supersonic at the narrowest place of the nozzle. In this problem we will assume a polytropic gas.
a) Explain why the sonic point has to lie at the narrowest point $A_{\rm min}$ of the funnel.

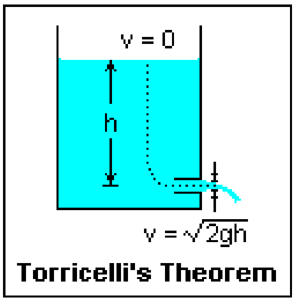

Figure 1: Torricelli's Theorem

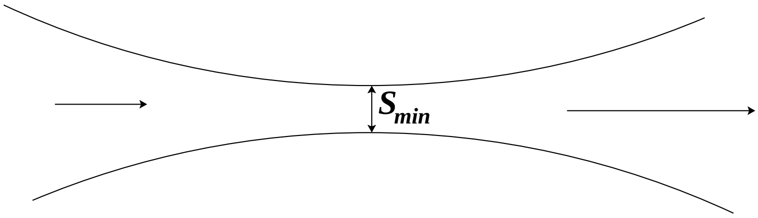

Figure 2: A De Laval nozzle

b) Use Bernouilli's principle to show that the sound speed at this sonic point has to be

$$c_* = c_0 \sqrt{2/(\gamma + 1)} \tag{3}$$

where c_0 is the sound speed at the start of the nozzle, where density and pressure are given by ρ_0 and p_0, and the velocity (close to) zero.
c) Justify the assumption of isentropic flow, and using this assumption show that the density at the sonic point has to be

$$\rho_* = \rho_0 \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma - 1)} \tag{4}$$

d) Using these results show that for the nozzle to work the mass flux through the pipe ($\dot{M} = \rho u A$) and the minimum cross section A_{min} have to be related by

$$A_{\mathrm{min}} = \frac{\dot{M}}{\sqrt{\gamma p_0 \rho_0}} \left(\frac{2}{\gamma + 1} \right)^{(\gamma+1)/(2-2\gamma)} \tag{5}$$

This condition is the equivalent of that on the mass loss rate in the case of a steady stellar wind.

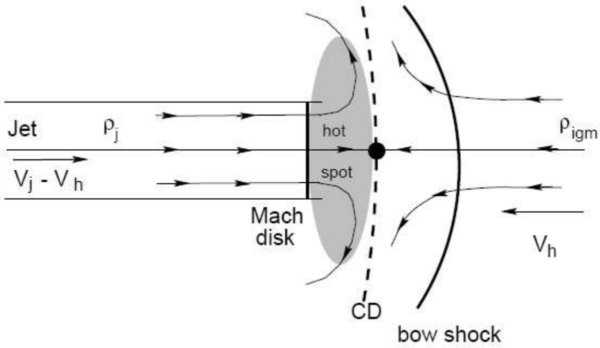

Figure 3: Schematic view of the structure of a radio jet.

15 The speed of radio jets

The radio lobes associated with powerful radio galaxies are caused by jets of material coming from the central black hole in the nucleus of the galaxy. The jet flow collides with the tenuous gas in the intergalactic medium (IGM). In this collision a structure forms as shown in Fig. 2. This structure contains the following:

- **Mach Disk**: a strong shock at the end of the jet
- **Hot Spot**: the region directly downstream from the Mach Disk with heated and compressed jet material. This region shows up clearly in radio maps due to the intense synchrotron radiation from relativistic electrons accelerated at the Mach Disk.
- **Contact Discontinuity (CD)**: the layer separating the shocked jet material from the shocked IGM gas.
- **Bow Shock**: shock preceding the Hot Spot and Mach Disk in the IGM. This bow shock forms because the whole structure of Mach Disk, Hot Spot and CD advance into the IGM at supersonic speeds.

The jet has a density ρ_j and a velocity v_j, the IGM has a density $\rho_{\rm IGM}$ and a velocity $u_{\rm IGM} = 0$ (both in the observer's frame).
a) Given that both the shock associated with the Mach Disk and the bow shock are assumed to be strong (i.e. Mach Number $\mathcal{M} \gg 1$), what is the typical density of the shocked jet material and of the shocked IGM material? Use an adiabatic index $\gamma = 5/3$.
Let us assume that the system of Mach Disk, CD and bow shock advances with a speed u_h into the (stationary) IGM. This speed is supersonic with respect to the IGM sound speed. We look at this system from a reference frame moving with u_h. The speed of

the jet material in this frame equals $u_j - u_h$ and the speed of the IGM equals $-u_h$. In this frame the entire flow pattern is stationary.
Consider the 'central flow lines' along the symmetry axis. Both the flow line originating in the jet and the flow line originating in the IGM end at the stagnation point where the velocity of both the jet material and IGM material vanishes. This stagnation point is indicated with a black dot in the figure.
Since the flow is stationary, the following quantity

$$\mathcal{E} = \frac{1}{2}u^2 + \frac{\gamma p}{(\gamma - 1)\rho} \tag{6}$$

is conserved along flow lines *and* across shocks
b) Argue why the preceding statement is true.
c) Use this conservation of $\mathcal{E}$ and the typical post-shock densities from a) to calculate the pressure on both sides of the stagnation point (i.e. in the shocked jet and IGM material), assuming the following

- The pre-shock density is ρ_j, and the jet speed is strongly supersonic so that $u_j - u_h \gg \sqrt{\gamma p_j / \rho_j)}$.
- The density in the IGM equals $\rho_{\rm IGM}$, and the advance speed of the radio lobe is strongly supersonic so that $u_h \gg \sqrt{\gamma p_{\rm IGM}/(\rho_{\rm IGM})}$.

d) At a contact discontinuity the densities may differ, but the pressures must be equal. Show that this condition at the stagnation point, together with the typical density calculated in a) determines the advance speed u_h as

$$u_h = \frac{u_j}{1 + \sqrt{\eta}} \tag{7}$$

with $\eta = \rho_{\rm IGM}/\rho_j$, the density ratio of the IGM and jet material.

16 Sonic point

a) A stellar wind is maintained at a temperature of $T = 2 \times 10^6$ K by magnetic heating; calculate the radius at which it achieves the isothermal sound speed if the star from which it blows has mass M. You may assume that the gas is atomic hydrogen. Evaluate your answer when M is the mass of the Sun $M_\odot = 2 \times 10^{30}$ kg.
b) For the case of an isothermal wind, assume there is an extra outward force working, of the mathematical form $f = Bu(\partial u/\partial r)$. An example of a force with such a dependence is radiation pressure due to optically thick spectral lines (Sobolev approximation). Show that the critical point is located at $r_{\rm c} = GM/(2c_s^2)$ (c_s the isothermal sound speed) and that the velocity there is *not* c_s, but equal to $c_s/\sqrt{1-B}$. This shows that the critical point and the sonic point in a stellar wind do not necessarily coincide.

17 Bondi accretion

Isothermal gas of pressure $\rho_0 c_1^2$ and density ρ_0 at large distances from a star is steadily accreted by this star of mass M.
a) Calculate the accretion rate assuming that the gas remains isothermal. At what radius does the infalling gas achieve the sound speed?
b) If $c_1 = 1$ km/s and $n_\infty = 10^9$ hydrogen molecules m^{-3} and the mass of the hydrogen atom is 1.66×10^{-27} kg, evaluate this radius in terms of the solar radius $R_\odot = 7 \times 10^5$ km.
c) Determine how long it will take a star of mass M to double its mass.