Astrophysical Gasdynamics: Exercises IV

December 10, 2007

18 Accretion disks

In the lectures we looked at the structure of an accretion disk by considering a continuity equation and the tangential velocity u_ϕ. Here we look at the two other velocities, u_R and u_z.
a) Write down the general equations for the evolution of u_R and u_z. Next write them for the conditions of a steady, axi-symmetric disk: $u_z = 0$, $\partial u_R/\partial t = 0$ and $\partial/\partial\phi = 0$.
b) Assume that the disk is thin, so that the scale height $H \ll R$ everywhere. Show that this leads to the result that $c_s \sim H\Omega$, where c_s is the sound speed, and Ω the angular velocity. The Keplerian speed at 1 AU around a 1 $M_\odot$ star is about 30 km/s. How cold does the disk need to be at that position if $H/R \sim 0.01$?
c) Show that the above also implies that the ratio of the pressure gradient and the gravitational force in the equation for u_R is very much smaller than 1. With this argue that the equation for u_R in fact shows us that the gas is approximately in Keplerian rotation.
d) If the accretion disk has a positive pressure gradient, what deviation does this cause on the rotation velocity of the gas? Solid bodies such as planetesimals and dust grains orbit in principle with Keplerian velocity. Why? If they experience a drag force from the gas, do you expect them to migrate in or out in the case of a positive pressure gradient in the gas? If the gas pressure has a local maximum, what do you expect the solids to do? And at a local minimum in gas pressure?
e) What accretion rate is needed for an accretion disc around a 1 $M_\odot$ protostar with a 1 $R_\odot$ radius to produce a luminosity of 1 $L_\odot$? Is it likely that a large part of the luminosity of such a protostar is due to accretion?

19 von Neumann Stability Analysis

The FTCS (Forward Time, Central Space) algorithm for the one-dimensional linear advection equation

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} = 0 \tag{1}$$

can be written as

$$\rho_j^{n+1} = \rho_j^n - u\frac{\Delta t}{2\Delta x}(\rho_{j+1}^n - \rho_{i-j}^n) \tag{2}$$

a) Using the technique of von Neumann stability analysis, show that this method is unconditionally unstable.
b) Rewrite the FTCS algorithm in conservative form, i.e. write it as

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}(F_{j-1/2} - F_{j+1/2}) \tag{3}$$

and find an expression for the flux function $F_{j+1/2}$.
The BTCS (Backward Time, Central Space) algorithm for the one-dimensional linear advection equation can be written as

$$\rho_i^{n+1} = \rho_i^n - u\frac{\Delta t}{2\Delta x}(\rho_{i+1}^{n+1} - \rho_{i-1}^{n+1})\,. \tag{4}$$

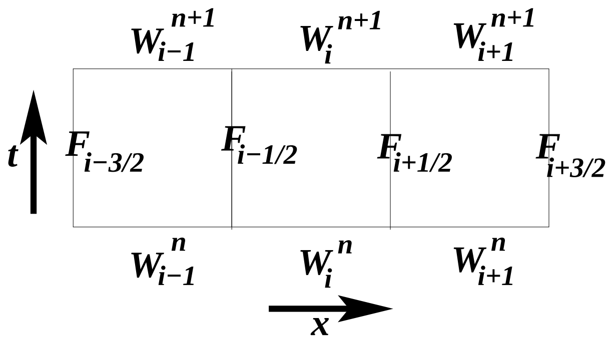
A method like this in which the fluxes depend on the solution for the next time t_{n+1} is called *implicit*.

c) Explain how can you may use such an implicit method in practice if the changes needed to calculate the new solution depend on the new solution itself?

d) Use the technique of von Neumann stability analysis to show that this method is unconditionally stable, i.e. there is not even a CFL condition. Explain why this is reasonable.

20 Conservation

We take the Euler equations for an isothermal gas ($\gamma = 1$), with isothermal sound speed $c_\mathrm{s} = 1$.
a) Argue that in the isothermal case we only need to consider the continuity and momentum equations.



Now consider three cells of a one-dimensional mesh, $j-1$, j, $j+1$. The cell size Δx is 1. Our numerical method uses some recipe to calculate the fluxes $\mathbf{F}$ $(\rho u, \rho u^2 + p)$ at the interfaces of these cells. The initial conditions at time t_n for the state vector $\mathbf{W} = (\rho, \rho u)^\mathrm{T}$ are as follows:

- (2, 0.6), (?, 1.2), (2, ?)

and at the end of time step at time $t_{\mathrm{n+1}} = t_n + \Delta t$ the state is

- (1.6, ?), (2.4, -0.1), (2.4, 1.96)

The fluxes for the four interfaces ($j - 3/2$ to $j + 3/2$) are

- (0.2, 1.04), (0.6, ?), (?, 3.48), (0.8, 2.32)

b) Find the missing values in the state and flux vectors (by considering the update of $\mathbf{W}^n$ to $\mathbf{W}^{n+1}$ with these fluxes $\mathbf{F}$).
c) Find the CFL number for the time step Δt.
d) From the numerical values given above, find the flux recipe used. Do you think this is a reasonable recipe for this problem? (think of the direction of the waves, i.e. the domain of influence).