Newton Versus Einstein

Prelude

We have four different natural forces: the weak, strong, electromagnetic and gravitational force. On cosmological scales, the dominant force determining the evolution of the universe is gravity.

Equivalence Principle

Newton: the shortest distance between two points is a straight line, the angles at the vertices of a triangle sum to $180^\circ \ldots \Rightarrow$ space is Euclidean. An object with no forces acting on it moves in a straight line at constant speed. Planets move on curved lines, with changing speeds because there is a force acting on them; gravity.

The force acting between two objects is

$$F = -\frac{GM_g m_g}{r^2},$$

where $m_g$ is the “gravitational mass”. The acceleration resulting from this gravitational force is

$$F = m_i a,$$

where $m_i$ is the “inertial mass”. The gravitational acceleration of an object toward a mass $M_g$ is

$$a = -\frac{GM_g}{r^2} \left( \frac{m_g}{m_i} \right),$$

which might vary from object to object. Observations show that the inertial and gravitational masses are the same to within 1 part in $10^{12}$.

$\Rightarrow$ The fact that the gravitational mass and inertial mass are equal is called the equivalence principle.

Thought experiment: Suppose you are sealed up inside an opaque hermetically sealed box, drop your teddy bear and find that it falls to the floor of
the box with an acceleration \( a = 9.8 \text{ m s}^{-2} \). The equivalence principle allows two possible interpretations, with no way of distinguishing between them:

1. The box is static, and a gravitational force is accelerating the bear downward.
2. No forces are acting on the bear, and the box is accelerating upward.

Imagine that a window in the side of the box opens to reveal that you are inside a spaceship which is being accelerated at 9.8 meters per second per second by a rocket engine. You grab a flashlight and shine a beam of light perpendicular to the direction of acceleration. Since the box is accelerating upward, the path of the light beam will appear to be bent downward, as seen by an observer in the box. Since the equivalence principle tells us that we should get the same result if the box is static in a gravitational field, we conclude that in the presence of gravity, photons follow a curved path. Since photons always follow the shortest path between two points, space is curved (not Euclidean).

General Relativity (GR) follows from the equivalence principle. In GR, mass and energy are interchangeable \((E = mc^2)\) and space and time form a four-dimensional space-time.

- Mass-energy tells space-time how to curve.
- Curved space-time tells mass-energy how to move.

The motion of the bear is explained by the bear following a geodesic ("the shortest path between two points") in curved space-time.

(Since the universe is isotropic and homogeneous on large scales, the curvature of space must also be isotropic and homogeneous on large scales.)

**Describing curvature**

A **metric** is a recipe for getting the distance between two points.

**Two dimensions**

**Flat (or Euclidean):** a plane which is infinite in area.

The angles of a triangle add to \( \pi \).

Metric (Pythagoras):

\[
 ds^2 = dx^2 + dy^2 ,
\]
or written in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 .$$

**Positive curvature:** the surface of a sphere which is finite in area.
The angles of a triangle add to \( \pi + A/R^2 \), where \( A \) is the area and \( R \) is the radius of the sphere.
Metric:

$$ds^2 = dr^2 + R^2 \sin^2(r/R)d\theta^2 .$$

**Negative curvature:** hyperboloids (or saddle shapes) which are infinite in area.
The angles of a triangle add to \( \pi - A/R^2 \).
Metric:

$$ds^2 = dr^2 + R^2 \sinh^2(r/R)d\theta^2 .$$

**Three dimensions**

**Flat:**

$$ds^2 = dx^2 + dy^2 + dz^2 ,$$
or

$$ds^2 = dr^2 + r^2 d\Omega^2 ,$$
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 .$$

**Positive curvature:**

$$ds^2 = dr^2 + R^2 \sin^2(r/R)d\Omega^2 .$$

**Negative curvature:**

$$ds^2 = dr^2 + R^2 \sinh^2(r/R)d\Omega^2 .$$

**General case:**

$$ds^2 = \frac{dx^2}{1 - \kappa x^2/R^2} + x^2 d\Omega^2 ,$$
where \( \kappa = 0, +1, -1 \) for flat, positive and negative case and

$$x = S_\kappa(r) = r, R\sin(r/R), R\sinh(r/R) .$$
Alternatively, we can write

\[ ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2. \]

Four dimensions (space-time)
In GR, time is incorporated into a space-time, where an event is defined as a part point in space and time. If we have no matter/gravity/curvature (this is special relativity), the distance between two events is defined as (the Minkowski metric)

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

or

\[ ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2. \]

Note that along the path of a light ray, \( ds^2 = 0 \).

General case: Including matter/gravity/curvature, we have (the Robertson-Walker metric)

\[ ds^2 = -c^2 dt^2 + a(t)^2 \left( \frac{dx^2}{1 - \kappa x^2 / R^2} + x^2 d\Omega^2 \right) = -c^2 dt^2 + a(t)^2 \left( dr^2 + S_\kappa(r)^2 d\Omega^2 \right), \]

where \( a(t) \) is the scale factor containing all information about the expansion (or contraction) of the universe.

Proper distance
The metrics give the distance between points. However, cosmological distances are difficult to measure for two reasons:
(1) They change with time.
(2) We can’t use a tape measure since distances are too large.

The proper distance \( d_P(t) \) between two points is the distance one would measure if I could freeze the expansion of the universe while stretching a tape measure from, e.g. here to a distant galaxy.
Putting us at the origin, we have

\[ ds = a(t)dr \]

and

\[ d_P(t) = a(t) \int_0^r dr = a(t)r . \]

The rate of change of the proper distance is

\[ \dot{d}_P \equiv v_P = \dot{a}r = \frac{\dot{a}}{a}d_P \equiv H_0d_P . \]

Note that \( v_P > c \) for \( d_P > c/H_0 \equiv d_H(t_0) \sim 4300 \) Mpc. \( d_H(t_0) \) is the Hubble distance.

As noted, it is impossible to measure \( d_P(t) \). What we can measure however, is redshifts. Imagine two successive wavecrests emitted at \( t_e \) and \( t_e + \lambda_e/c \) from a galaxy at \( r \) and received at \( t_0 \) and \( t_0 + \lambda_0/c \):

\[
\begin{align*}
  c \int_{t_e}^{t_0} \frac{dt}{a(t)} &= \int_0^r dr = r, \\
  c \int_{t_e + \lambda_e/c}^{t_0 + \lambda_0/c} \frac{dt}{a(t)} &= \int_0^r dr = r .
\end{align*}
\]

\[
\Rightarrow
\begin{align*}
  \int_{t_e}^{t_0} \frac{dt}{a(t)} &= \int_{t_e + \lambda_0/c}^{t_0 + \lambda_0/c} \frac{dt}{a(t)} \\
  \Rightarrow
  \int_{t_e}^{t_e + \lambda_e/c} \frac{dt}{a(t)} &= \int_{t_0}^{t_0 + \lambda_0/c} \frac{dt}{a(t)} \\
  \Rightarrow
  \frac{\lambda_e}{a(t_e)} &= \frac{\lambda_0}{a(t_0)} \\
  \Rightarrow
  \frac{\lambda_0}{\lambda_e} = \frac{\nu_e}{\nu_0} = 1 + z = \frac{dt_0}{dt_e} = \frac{a(t_0)}{a(t_e)} .
\end{align*}
\]

**Summary**

- Gravity is the most important force on cosmological scales.

- In GR, mass curves space-time and curvature determines the motion of mass. Note however that in most cases, the GR and Newton approach are interchangeable.

- The redshift of an object tells us what the scale-factor of the universe was at the time of emission.