Cosmic Dynamics

Prelude

\[ ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dx^2}{1 - \kappa x^2/R_0^2} + x^2 d\Omega^2 \right] = -c^2 dt^2 + a(t)^2 \left[ dr^2 + S_\kappa(r)^2 d\Omega^2 \right] \]

and

\[ x = S_\kappa(r) = r, R_0 \sin(r/R_0), R_0 \sinh(r/R_0) . \]

In a homogeneous and isotropic universe, everything is contained in \( \kappa, R_0 \) and the scale factor \( a(t) \). If space is curved, \( \kappa \) is non-zero and \( R_0 \) is the radius of curvature at \( t = t_0 \). The scale factor \( a(t) \) is a dimensionless number, normalized to unity at \( t = t_0 \).

The Friedmann equation

The equation that describes how the scale factor evolves with time is known as the Friedmann equation.

Newtonian derivation:

The mass density in a homogeneous, isotropic universe, \( \rho(t) \), depends on time, but not on position. Draw a sphere about the origin with constant mass

\[ M_S = \frac{4\pi}{3} \rho(t) R_S(t)^3 . \]

We can write the physical radius \( R_S(t) \) in the form

\[ R_S(t) = a(t) r_S , \]

where \( a(t) \) is the scale factor of the universe and \( r_S \) is the radius as measured at \( t = t_0 \) (the comoving radius).

A test mass, \( m \) located outside a spherically symmetric object experiences the same force it would feel if all mass was concentrated at the object’s center.
Also, the test mass experiences no gravitational force from the matter outside the sphere. Thus,

\[ F = -\frac{GM_sm}{R_S(t)^2}. \]

The gravitational potential energy of the test mass (per unit mass) is

\[ E_{\text{pot}} = \frac{GM_S}{R_S(t)} = -\frac{4\pi}{3}Gr_S^2\rho(t)a(t)^2, \]

and the kinetic energy (per unit mass)

\[ E_{\text{kin}} = \frac{1}{2} \frac{dR_S(t)^2}{dt} = \frac{1}{2} r_S^2 \dot{a}^2. \]

Since the sum of the gravitational potential energy and the kinetic energy is constant, \( U \equiv E_{\text{kin}} + E_{\text{pot}} \) \( \Rightarrow \)

\[ \frac{1}{2} r_S^2 \dot{a}^2 = \frac{4\pi}{3} Gr_S^2\rho(t)a(t)^2 + U, \]

\[ \Rightarrow \]

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{2U}{r_S^2 a(t)^2}. \]

The future of the expanding sphere depends on the value of \( U \):
- \( U > 0 \) \( \Rightarrow \) \( \dot{a}^2 \) always positive \( \Rightarrow \) eternal expansion.
- \( U < 0 \) \( \Rightarrow \) \( \dot{a}^2 \) zero at \( a_{\text{max}} = -(GM_S)/(Ur_S) \) \( \Rightarrow \) contraction.
- \( U = 0 \) \( \Rightarrow \) \( \dot{a}^2 \) goes to zero in the infinite future.

The exact form of the Friedmann equation, as derived by Friedmann himself using GR, is

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \epsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a^2}. \]

The mass density \( \rho \) has been replaced by an energy density \( \epsilon \) divided by \( c^2 \) (e.g., photons also contributes to \( \epsilon \)). The second substitution is

\[ \frac{2U}{r_S^2} = -\frac{\kappa c^2}{R_0^2}. \]

That is, if the energy \( U \) for a test mass is zero, the universe is spatially flat. If the test mass is unbound \( (U > 0) \), the universe is negatively curved. If the test mass is bound \( (U < 0) \), the universe is positively curved.
Since $H(t) = \dot{a}/a$, we can write

$$H(t)^2 = \frac{8\pi G}{3c^2} \epsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}.$$  

The value of $H(t)$ today [measured from $v(t) = H_0 d(t)$] is

$$H_0 \equiv H(t_0) \sim 70 \text{ km s}^{-1} \text{ Mpc}^{-1}.$$ 

$H(t)$ is called the “Hubble parameter” and the current value of the Hubble parameter, $H_0$, is called the “Hubble constant”.

For a given value of the Hubble parameter, there is a critical density,

$$\epsilon_c(t) \equiv \frac{3c^2}{8\pi G} H(t)^2.$$ 

If the energy density of the universe is greater than this value, the universe will be positively curved. If the energy density is less than this value, the universe will be negatively curved. The current value of the critical density is

$$\epsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2 \sim 5000 \text{ MeV m}^{-3},$$ 

or as an equivalent mass density,

$$\rho_{c,0} \equiv \epsilon_{c,0}/c^2 \sim 9 \times 10^{-27} \text{ kg m}^{-3} \sim 1.4 \times 10^{11} \text{ M}_\odot \text{ Mpc}^{-3},$$

roughly equivalent to a density of one hydrogen atom per 200 liters or one galaxy per Mpc cube.

Define a dimensionless density parameter as the ratio of the energy density and the critical density

$$\Omega(t) \equiv \frac{\epsilon(t)}{\epsilon_c(t)},$$

⇒ we can write

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2} \frac{1}{H(t)^2}.$$  

Note that since $\kappa$ is constant, $1 - \Omega(t)$ does not change sign.

Directly measuring the curvature by geometric methods is difficult. Consider
a positively curved universe with circumference \( C_0 = 2\pi R_0 \). If \( C_0 \ll c t_0 \sim c/H_0 \), photons will have had time to circumnavigate the universe several times (in the past, \( C_0 \) was even smaller). Put \( C_0 \) to 10 million light years. Looking toward M31, which is 2 million light years away, we would see one image 2 million light years away, showing M31 as it was 2 million years ago, another image 12 million light years away, showing M31 as it was 12 million years ago, and so on (and one image in the opposite direction showing M31 as it was 8 million years ago etc.). Since we don’t see periodicities of this sort, we conclude that if the universe is positively curved, \( R_0 > c/H_0 \approx 4300 \) Mpc. There is a lower limit on the radius of curvature also for a negatively curved universe. From the Friedmann equation as evaluated at the present moment,

\[
H_0^2 = \frac{8\pi G}{3c^2} \epsilon_0 - \frac{k c^2}{R_0^2},
\]

we again find that \( R_0(\text{min} = c/H_0) \).

**The Fluid and Acceleration equations**

Energy conservation tells us that

\[
dQ = dE + PdV,
\]

where \( dQ \) is the heat flow in or out of a region, \( dE \) is the change in the internal energy, \( P \) is the pressure and \( dV \) is the change in volume of the region. In a homogeneous universe, \( dQ = 0 \) and

\[
\dot{E} + PV = 0.
\]

A sphere with radius \( R_S(t) = a(t)r_S \) has

\[
V(t) = \frac{4\pi}{3} r_S^3 a(t)^3,
\]

and internal energy

\[
E(t) = V(t)\epsilon(t).
\]

The rate of change of the internal energy is

\[
\dot{E} = V\dot{\epsilon} + \dot{V}\epsilon = V \left( \dot{\epsilon} + 3\frac{\dot{a}}{a} \right),
\]
giving
\[ \dot{\epsilon} + 3 \frac{\dot{a}}{a} (\epsilon + P) = 0 , \]
⇒ the fluid equation.

The Friedmann and fluid equations can be combined to derive the acceleration equation:
\[ \ddot{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3P) . \]
The energy density \( \epsilon \) slows down the expansion. A component with a pressure \( P < -\epsilon/3 \) will cause the expansion of the universe to speed up. A cosmological constant has \( P = -\epsilon \). (Negative pressure is the same as tension.)

**Learning to love Lambda**

Einstein first published GR in 1915 when the universe was thought to be static. (In fact, it was by no means settled that galaxies besides our own actually existed.) The only permissible static universe is a totally empty universe.

Einstein introduced a new term, the **cosmological constant** \( \Lambda \), into the Friedmann equation
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} . \]
The fluid equation is unchanged and the acceleration equation takes the form
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3P) + \frac{\Lambda}{3} . \]
⇒ Introducing \( \Lambda \) is equivalent to introducing an energy density
\[ \epsilon_\Lambda = \frac{c^2}{8\pi G} \Lambda . \]
For \( \epsilon_\Lambda \) to remain constant with time, the associated pressure must be
\[ P_\Lambda = -\epsilon_\Lambda = -\frac{c^2}{8\pi G} \Lambda . \]
Now, setting $\Lambda = 4\pi G \rho$, the Friedmann equation reduces to

$$0 = \frac{4\pi G}{3} \rho - \frac{\kappa c^2}{R_0^2},$$

i.e., a positively curved, static universe. However, the solution is unstable and Einstein was eager to abandon $\Lambda$ as he was presented with Hubble’s 1929 evidence for an expanding universe.

Note however that Hubble’s value of $H_0 \sim 500 \text{ km s}^{-1} \text{ Mpc}^{-1}$ leads to a Hubble age of $H_0^{-1} \sim 2 \text{ Gyr}$, less than half the age of the Earth. If the value of $\Lambda$ is large enough to make $\ddot{a} > 0$, then $\dot{a}$ was smaller in the past than it is now, and consequently the universe is older than the Hubble time. In fact, you can make the universe arbitrarily old by cranking up the value of $\Lambda$. More recently, $\Lambda$ has been employed to accelerate the expansion rate of the universe.

What is the cosmological constant? The Heisenberg uncertainty principle $\Delta E \Delta t \leq \hbar$ permits particle-antiparticle pairs to spontaneously appear and then annihilate in an otherwise empty vacuum. The energy density $\epsilon_{\text{vac}}$ is associated with the density of the virtual particles and antiparticles

$$\epsilon_{\text{vac}} \sim \frac{E_p}{E_p} \sim 3 \times 10^{133} \text{ eV/ m}^3.$$

This is 124 orders of magnitude larger than the critical density for our universe $\Rightarrow$ a spectacularly poor match between theory and observation.

**Equations of State**

Since we have two independent equations which describe how the universe expands and three unknowns – the functions $a(t)$, $\epsilon(t)$, and $P(t)$ – we need an equation of state, relating the pressure $P$ to the energy density $\epsilon$. For all substances of cosmological importance, the equation of state can be written in a simple linear form:

$$P = w \epsilon,$$

where $w$ is a dimensionless constant. A low-density gas of non-relativistic particles has

$$w \sim \frac{\langle v^2 \rangle}{3c^2} \ll 1.$$
For a gas of radiation (either photons or highly relativistic massive particles)

\[ w = \frac{1}{3}. \]

(Small perturbations in the pressure will travel at the speed of sound

\[ c_s = c |\frac{dP}{d\epsilon}|^{1/2} = \sqrt{|w|}c. \]

To preserve causality, \( w \) is restricted to the range \(-1 \leq w \leq 1.\)

The particularly interesting values of \( w \) are:

- \( w = 0 \) (cold, non-relativistic gas or dust)
- \( w = 1/3 \) (radiation)
- \( w = -1 \) (cosmological constant)

(A network of cosmic strings has \( w = -1/3 \) and that a network of domain walls has \( w = -2/3.\))

**Summary**

The **Friedmann equation** is given by

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2a(t)^2}.
\]

The **fluid equation** is given by

\[
\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0.
\]

The **acceleration equation** is given by

\[
\frac{\ddot{a}}{a} = -\frac{4\pi Gq}{3c^2}(\epsilon + 3P).
\]

Given an **equation of state** for the different components and the current energy densities, we can solve for \( a(t) \) and \( \epsilon(t) \).