Notes for Cosmology course, fall 2005

Single-Component Universes

Prelude

The mathematical relation among $\epsilon(t)$, $P(t)$, and $a(t)$ is given by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{\kappa c^2}{R_0^2 a^2} \ ,$$

the fluid equation,

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \ ,$$

and the equation of state,

$$P = w\epsilon \ .$$

We will begin by solving the equations with only one term on the RHS of the Friedmann equation.

Evolution of energy density

The energy density and pressure for the different components are additive.

$$\epsilon_{\rm tot} = \sum_w \epsilon_w \ .$$

$$P_{\rm tot} = \sum_w P_w = \sum_w w\epsilon_w \ ,$$

$\Rightarrow$ the fluid equation holds for each component separately.

For a component of the universe with equation-of-state parameter w (w constant),

$$\dot{\epsilon}_w + 3\frac{\dot{a}}{a}(\epsilon_w + P_w) = 0 \ ,$$

$\Rightarrow$

$$\frac{d\epsilon_w}{\epsilon_w} = -3(1+w)\frac{da}{a} \ ,$$

$\Rightarrow$

$$\epsilon_w(a) = \epsilon_{w,0} a^{-3(1+w)} \ .$$

Non-relativistic particles ($v \ll c$) or ordinary matter, dust $\Rightarrow w = 0$
$\epsilon_m = \epsilon_{m,0}/a^3$

Relativistic particles ($v \sim c$) or radiation $\Rightarrow w = 1/3$
$\epsilon_r = \epsilon_{r,0}/a^4$

Cosmological constant $\Rightarrow w = -1$
$\epsilon_\Lambda = \text{const}$

Note that the curvature term in the Friedmann equation is proportional to a^{-2}.

Explanation:
The energy density in Λ is by definition constant.
The number density of particles has the dependence $n \propto a^{-3}$. The energy of a non-relativistic particle is contributed almost entirely by its rest mass, $E = mc^2$. The energy of a photon, however, has the dependence $E = hc/\lambda \propto a^{-1}$, since the wavelength of light expands along with the global expansion of the universe. Thus, for photons or other relativistic particles, $\epsilon = nE \propto a^{-3}a^{-1} \propto a^{-4}$. We assume that photons are neither created nor destroyed. This is true since photons from stars contribute less than 10 % of the total radiation density.

If the universe contains different components, in the limit $a \to 0$, the component with the largest value of w was dominant. When $a \to \infty$, the component with the smallest value of w will be dominant. Thus, in a continuously expanding universe containing radiation ($w = 1/3$), matter ($w = 0$), and a cosmological constant ($w = -1$), at early times the radiation is dominant and at late times the cosmological constant is dominant.

The radiation density is known from CMB observations ($\epsilon_r \sim \alpha T_0^4$). The matter density is about 30% of the critical density $\Rightarrow$

$$\frac{\epsilon_{m,0}}{\epsilon_{r,0}} \approx 3600 \ ,$$

$\Rightarrow$ non-relativistic matter is currently "dominant" over radiation. However, in the past the ratio of the densities was

$$\frac{\epsilon_m(a)}{\epsilon_r(a)} = \frac{\epsilon_{m,0}}{\epsilon_{r,0}} a \ .$$

$\Rightarrow$ the energy densities were equal at

$$a = a_{\rm rm} = \frac{\epsilon_{r,0}}{\epsilon_{m,0}} \sim 2.8 \times 10^{-4} \ .$$

Today,

$$\frac{\epsilon_\Lambda}{\epsilon_{m,0}} \sim \frac{0.7}{0.3} \ ,$$

$\Rightarrow$ Λ is dominant today. In the past

$$\frac{\epsilon_\Lambda}{\epsilon_m(a)} = \frac{\epsilon_\Lambda}{\epsilon_{m,0}} a^3 \ ,$$

$\Rightarrow$

$$a = a_{\rm m\Lambda} = \left(\frac{\epsilon_{m,0}}{\epsilon_\Lambda}\right)^{1/3} \sim 0.75 \ .$$

In a continuously expanding universe, we can use the scale factor a or the redshift z instead of time t. The conversion from a or z to time t is not trivial in a multiple-component universe. The solution to

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_w \epsilon_{w,0} a^{-1-3w} - \frac{\kappa c^2}{R_0^2} \ ,$$

has a simple analytic form in the case when there is a single term on the right hand side.

Curvature only

A particularly simple universe is one which is empty

$$\dot{a}^2 = -\frac{\kappa c^2}{R_0^2} \ .$$

One solution to this equation has $\dot{a} = 0$ and $k = 0$ $\Rightarrow$ an empty, static, spatially flat universe (Minkowski space).

We can also have $\kappa < 0$, i.e., a negatively curved empty universe with

$$\dot{a} = \pm \frac{c}{R_0}, \quad a = \frac{t}{t_0} \ .$$

That is, if the energy density is zero, then there are no gravitational forces at work and the relative velocity of any two points is constant (c.f. Newton).

Flat universes

For a flat ($\kappa = 0$), single-component universe, the Friedmann equation takes the form

$$\dot{a}^2 = \frac{8\pi G \epsilon_0}{3c^2} a^{-1-3w} \ .$$

Ansatz: $a \propto t^q \ \Rightarrow q = \frac{2}{3(1+w)}$ ([$w > -1$]) and

$$a(t) = (t/t_0)^{2/(3+3w)} \ .$$

Generally, in a flat, single-component universe, the energy density always decreases as $1/t^2$ (as long as $w > -1$) and

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+w)} t_0^{-1} \ .$$

The age of the universe (t_0) is linked to the Hubble time (H_0^{-1}) by the relation

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} \ ,$$

$\Rightarrow$ if $w > -1/3$, the universe is *younger* than the Hubble time; if $w < -1/3$, the universe is *older* than the Hubble time.

This can also be understood by considering the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(1 + 3w)\epsilon \ .$$

If $w > -1/3$, then $\ddot{a}$ is negative and the expansion speeds were faster in the past then they are now $\Rightarrow$ the universe is younger than H_0^{-1}.
If $w > -1/3$, then $\ddot{a}$ is positive and the expansion speeds were slower in the past than they are now $\Rightarrow$ the universe is older than H_0^{-1}.

Matter only

In a flat universe with matter only, the scale factor is

$$a_m(t) = (t/t_0)^{2/3} \ ,$$

and $t_0 = (2/3)H_0^{-1}$.

Radiation only

In a flat universe with radiation only, the scale factor is

$$a_r(t) = (t/t_0)^{1/2} \ ,$$

and $t_0 = (1/2)H_0^{-1}$.

Lambda only

The Friedmann equation for a flat, Λ-dominated universe is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon_\Lambda = \text{constant} \ .$$

This can be written as

$$\dot{a} = H_0 a \ ,$$

where the Hubble constant is truly constant. The solution is

$$a(t) = e^{H_0(t-t_0)} \ ,$$

i.e. exponentially expanding (c.f., the "steady state universe").

A steady state universe maintains a constant density by the method of having matter continuously created. A Λ-dominated universe maintains a constant density by having virtual particles and antiparticles pop out of the vacuum and swiftly reannihilate.

Summary

Different components of the universe dominate at different epochs, thus it's a fair beginning to solve the Friedmann equation with only one term on the RHS. The solutions are all monotonically increasing (or decreasing) scale factors.

In order to have more interesting universes (as our own), we need to consider multiple component universes.