Multi-Component Universes

Prelude

The Friedmann equation is given by

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2} \ ,$$

where the energy density associated with Λ is

$$\epsilon_\Lambda = \frac{c^2}{8\pi G}\Lambda \ .$$

Defining $\Omega_0 = \Omega_{\rm tot} = \epsilon_0/\epsilon_{c,0}$ where

$$\epsilon_{c,0} \equiv \frac{3c^2 H_0^2}{8\pi G} \ .$$

and using

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \ ,$$

we can write

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} - \frac{1-\Omega_0}{a^2}$$

or

$$\frac{da}{dt} = H_0\left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right]^{1/2} \ .$$

We can then find the age of the universe (t) as a function of the scale factor a by performing the integral

$$H_0 t = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a - (1-\Omega_0) + \Omega_{\Lambda,0}a^2]^{1/2}} \ .$$

In the general case, this must be solved numerically.

Matter + Curvature

Consider a spatially curved universe whose energy density is due entirely to non-relativistic matter ($\Omega_{m,0} = \Omega_0$): $\epsilon = \epsilon_m = \epsilon_0/a^3 \;\Rightarrow$ the Friedmann equation

$$\frac{H^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2} \ ,$$

$\Rightarrow$ the universe will cease to expand if $\Omega_0 > 1$ ($\kappa > 0$). At the time of maximum expansion,

$$H(t) = 0 = \frac{\Omega_0}{a_{\max}^3} + \frac{1-\Omega_0}{a_{\max}^2} \ ,$$

and

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \ .$$

The contraction phase is just the time-reversal of the expansion phase. *That the contraction is a time-reversal of the expansion is only true on large scales.*

If $\Omega_0 < 1$, *both* terms on the right hand side are greater than zero $\Rightarrow$ if the universe is expanding right now it will continue to expand forever. At early times, the matter term of the Friedmann equation will dominate, and the universe will expand like a flat, dust-dominated universe, with $a \propto t^{2/3}$. At later times, the curvature term dominates, and the universe will expand like an empty, negatively curved universe, with $a \propto t$.

Since

$$\frac{da}{dt} = H_0 \left[\frac{\Omega_0}{a} + (1-\Omega_0)\right]^{1/2} \ ,$$

the age t of the universe at a given scale factor a is

$$H_0 t = \int_0^a \frac{da}{[\Omega_0/a + (1-\Omega_0)]^{1/2}} \ .$$

A plot of a versus t for a universe with $\Omega_0 = 0.9$ is shown on the attached graph, along with the solution for $\Omega_0 = 1.1$ and $\Omega_0 = 1$. Note that although the ultimate fates of these universes are very different from each other, it is very difficult, at the present moment, to distinguish them.

We have the following cases:

$\Omega_0 = 1$ ($\kappa = 0$): **Big Chill** ($a \propto t^{2/3}$) and

$$t_0 = \frac{2}{3} H_0^{-1} \qquad [\Omega_0 = 1] \ .$$

$\Omega_0 < 1$ ($\kappa < 0$): **Big Chill** ($a \propto t$) and

$$t_0 \approx H_0^{-1} \qquad [\Omega_0 \ll 1] \ .$$

$\Omega_0 > 1$ ($\kappa > 0$,): **Big Crunch** and

$$t_{\rm crunch} \approx \pi H_0^{-1} \Omega_0^{-1/2} \qquad [\Omega_0 \gg 1] \ .$$

Matter + Lambda

Consider a flat universe in which the matter density is $\Omega_{m,0}$ and the density in a cosmological constant $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ $\Rightarrow$ the Friedmann equation

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}) \ .$$

If $\Omega_{\Lambda,0} = 1 - \Omega_{m,0} < 0$, the universe ceases to expand ($H = 0$) at a maximum scale factor

$$a_{\rm max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1} \right)^{1/3} ,$$

and the universe crunches back down to infinite density in time

$$t_{\rm crunch} = \frac{2\pi}{3 H_0 \sqrt{\Omega_{m,0} - 1}} \ .$$

A flat universe with a positive cosmological constant will continue to expand forever if it is expanding now.

The Friedmann equation can be integrated analytically

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln \left[(a/a_{m\Lambda})^{3/2} + \sqrt{1 + (a/a_{m\Lambda})^3} \right] ,$$

assuming $\Omega_{m,0} < 1$.

Matter + Lambda with $\Omega_{m,0} = 0.3$ and $\Omega_\Lambda = 0.7$ is a good fit to current data $\Rightarrow$ the density terms will be equal at scale factor

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \sim \left(\frac{0.3}{0.7}\right)^{1/3} \approx 0.75 \ .$$

corresponding to a redshift

$$z_{m\Lambda} = \frac{1}{a_{m\Lambda}} - 1 \approx 0.33 \ .$$

The age at which the energy density were equal was

$$t_{m\Lambda} = 0.702 H_0^{-1} = 9.8 \pm 1.0\,\mathrm{Gyr} \ .$$

The current age of the universe is

$$t_0 = 0.964 H_0^{-1} = 13.5 \pm 1.3\,\mathrm{Gyr} \ .$$

In the far past, the universe was dominated by matter and $a \propto t^{2/3}$. In the far future, the universe will be dominated by Λ, and $a(t) \propto e^{\sqrt{1-\Omega_{m,0}}H_0 t}$.

Matter + Curvature + Lambda

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{1-\Omega_{m,0}-\Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0} \ .$$

A particular set of model universes are positively curved universes containing matter and a positive cosmological constant, e.g. Einstein's static universe with $\kappa = +1$ and $\epsilon_\Lambda = \epsilon_m/2$.

For some choices of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, the value of H^2 will be positive for small values of a (where matter dominates) and for large values of a (where Λ dominates), but negative for intermediate values of a, where curvature dominates. Since negative values of H^2 are unphysical, this implies that some universes have a forbidden range of scale factors $\Rightarrow$ a beginning contraction will stop at some minimum value $a_{\min}$ and then expand outward again in a "Big Bounce".

If the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are fine tuned the universe begins by expanding with $a \propto t^{2/3}$ after which it enters a "loitering" phase, in which a is nearly constant for a long period of time (c.f. Einstein's static universe). After a period of slow growth, the cosmological constant takes over, and the universe enters a period of exponential expansion. There's strong observational evidence that we don't live in a "Big Bounce" or "loitering" universe, e.g., the "loitering" redshift, blueshifts, maximal redshifts etc (and the CMB!).

Radiation + Matter

If $\Omega_{m,0} \approx 0.3$ and $\Omega_{r,0} = 8.4 \times 10^{-5}$, then the contributions of radiation and matter were equal at a scale factor

$$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.8 \times 10^{-4} \ ,$$

corresponding to a redshift $z_{rm} \approx 3600$ and a $t_{rm} \sim 47\,000$ years. Since radiation is dominant for only a short period, our calculation of $t_0 = 13.5\,\mathrm{Gyr}$ is still OK!

Benchmark model

A flat universe with radiation (photons and neutrinos), matter (baryons and dark matter) and a cosmological constant (???).

$H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

$\Omega_{r,0} \sim 10^{-4}$ dominant at $3600 < z < \infty \ \Rightarrow a \propto t^{1/2}$

$\Omega_{m,0} \sim 0.3$ dominant at $1/3 < z < 3600 \ \Rightarrow a \propto t^{2/3}$

$\Omega_{\Lambda,0} \sim 0.7$ dominant at $0 < z < 1/3 \ \Rightarrow a \propto \exp Kt$

The current age of the universe is ~ 13.5 Gyr.

(WMAP + HST $\Rightarrow$ 13.7 Gyr, WMAP only $\Rightarrow$ 13.4 Gyr.)

Summary

Different components of the universe dominate at different epochs allowing for simple expressions for the scale factor at limited time intervals. However, including radiation, matter, curvature and Λ allows for Big Chills, Big Crunches, loitering universes...

It seems that we live in a flat Big Chill universe with $\Omega_{m,0} \sim 0.3$, $\Omega_{\Lambda,0} \sim 0.7$ and $\Omega_{r,0} \sim 10^{-4}$.