Measuring Cosmological Parameters

Introduction

We would like to know the functional form of $a(t)$, from the Big Bang ($t = 0$) into the indefinite future (or until the Big Crunch, whichever comes first). If we knew the expansion rate today (as given by the Hubble constant) and the energy density of each component, we would be able to compute $a(t)$.

In reality, if only we could determine $a(t)$ from observations, we would then know $\epsilon$ for each component.

$H_0$ and $q_0$

We don’t know the exact form of $a(t) \Rightarrow$ do a Taylor expansion about the point of minimum ignorance. The expansion of a function $f(x)$ around a point $x_0$ is

$$f(x) = f(x_0) + \frac{df}{dx}\bigg|_{x=x_0} (x-x_0) + \frac{1}{2} \frac{d^2f}{dx^2}\bigg|_{x=x_0} (x-x_0)^2 + \ldots .$$

To exactly reproduce an arbitrary function $f(x)$ for all values of $x$, an infinite number of terms is required in the expansion. Using only the first few terms of the expansion gives a good approximation in the vicinity of $x_0$ if $f$ doesn’t fluctuate wildly with $x$.

The scale factor can be approximated as

$$a(t) \approx a(t_0) + \dot{a}\big|_{t=t_0} (t-t_0) + \frac{1}{2} \ddot{a}\big|_{t=t_0} (t-t_0)^2 ;$$

or

$$a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 ,$$

where

$$H_0 \equiv \frac{\dot{a}}{a}\bigg|_{t=t_0} .$$
and $q_0$ is the **deceleration parameter**

$$q_0 \equiv -\left(\frac{\ddot{a}}{a^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}.$$  

$q_0$ positive $\Rightarrow$ the expansion is currently slowing down.  
$q_0$ negative $\Rightarrow$ the expansion is currently speeding up.

$H_0$ and $q_0$ tell us approximately how the universe expands. From the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3P),$$

we have

$$q_0 = \frac{1}{2} \sum_w \Omega_{w,0}(1 + 3w).$$

For $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$ we have

$$q_0 = \frac{1}{2} (0.3) - 0.7 = -0.55,$$

i.e an accelerating universe.

**Measuring $H_0$:** In the limit $z \ll 1$, Hubble’s law is

$$cz = H_0 d,$$

i.e. measure the redshift $z$ and distance $d$ for a large sample of galaxies and fit a straight line to the plot – the slope of the line gives you $H_0/c$. But, distances are hard to measure (and define)!

The proper distance is

$$d_P(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)}.$$

Writing

$$\frac{1}{a(t)} \approx 1 - H_0(t - t_0) + \frac{1 + q_0}{2} H_0^2 (t - t_0)^2,$$

we get

$$d_P(t_0) \approx c(t_0 - t_e) + \frac{cH_0}{2} (t_0 - t_e)^2.$$

The first term is what the proper distance would be in a static universe; the second term is a correction due to the expansion.
Since we measure redshifts, we use

\[ z \approx H_0 (t_0 - t_e) + \frac{1 + q_0}{2} H_0 (t_0 - t_e)^2 , \]

and (inverting the relation)

\[ t_0 - t_e \approx H_0^{-1} \left[ z - \frac{1 + q_0}{2} z^2 \right] . \]

We can now write \( d_P(t_0) \) as a function of its observed redshift:

\[ d_P(t_0) \approx \frac{c}{H_0} \left[ z - \frac{1 + q_0}{2} z^2 \right] + \frac{cH_0}{2} \frac{z^2}{H_0^2} \approx \frac{c}{H_0} z \left[ 1 - \frac{1 + q_0}{2} z \right] . \]

If we can measure both \( z \) and \( d_P(t_0) \), we can get \( H_0 \) and \( q_0 \)!

**Measuring distances**

Within our Solar System, astronomers measure the distance to the Moon and the planets by bouncing radar signals off them ⇒ useful to distances of \( \sim 10 \text{ AU} \).

Distances to stars within our galaxy are measured using the method of stellar parallax. The **parallax distance** \( d_\pi \) is

\[ d_\pi = 1 \text{ pc} \left( \frac{b}{1 \text{ AU}} \right) \left( \frac{\delta \theta}{1 \text{ arcsec}} \right)^{-1} . \]

A star at a distance of 1 kiloparsec has a parallax of 1 milliarcsecond, using the Earth’s orbit as a baseline. This is the limit of what currently can be measured ⇒ local methods inapplicable on cosmological scales.

**Luminosity distance**

A **standard candle** is an object with known luminosity \( L \). If we know the luminosity \( L \) and the measured flux \( f \) of an object is, we can define the **luminosity distance**

\[ d_L \equiv \left( \frac{L}{4\pi f} \right)^{1/2} . \]
In a static, Euclidean universe, the luminosity distance and the proper distance are identical (since \( f = L/A = L/(4\pi d^2) \)). In a universe which is curved and/or expanding, the distances are not the same.

Start with the Robertson-Walker metric:

\[
ds^2 = -c^2 dt^2 + a(t)^2 \left[ dr^2 + S_\kappa(r)^2 d\Omega^2 \right]
\]

where

\( x = S_\kappa(r) = r, R \sin(r/R), R \sinh(r/R) \)

for \( \kappa = 0, +1, -1 \). A standard candle of luminosity \( L \) emits photons at a time \( t_e \) which at \( t_0 \) are spread over a sphere of proper radius \( d_P(t_0) \) and surface area \( A_P(t_0) = 4\pi S_\kappa(r)^2 \). In addition to this geometric effect, the observed flux will be decreased by a factor of \((1 + z)^{-2}\). First, the expansion of the universe causes the energy of each photon from the standard candle to decrease

\[
E_0 = \frac{E_e}{1 + z}.
\]

In addition, if two photons are emitted separated by a time interval \( \delta t_e \), by the time we observe them, the interval will have grown to \( \delta t_0 = (1 + z)\delta t_e \) at observation \( \Rightarrow \)

\[
f = \frac{L}{4\pi S_\kappa(r)^2(1 + z)^2},
\]

\( \Rightarrow \)

\[
d_L = S_\kappa(r)(1 + z).
\]

**Angular-diameter distance**

Suppose that you have, instead of a standard candle, a **standard yardstick**, defined as a class of objects whose proper length \( \ell \) is the same. If a standard yardstick is aligned perpendicular to our line of sight and we measure an angular size \( \delta \theta \), we can define an **angular size distance**:

\[
d_A \equiv \frac{\ell}{\partial \theta}.
\]

The distance between the two ends of the yardstick at time of emission will be

\[
l = ds = a(t_e)S_\kappa(r)\delta \theta = \frac{S_\kappa(r)\delta \theta}{1 + z},
\]
\[ d_A \equiv \frac{\ell}{\delta \theta} = \frac{S_\delta(r)}{1 + z} = \frac{d_L}{(1 + z)^2}. \]

Note that in the limit that \( z \to 0 \)

\[ d_A \approx d_L \approx d_{P,0} \approx \frac{c}{H_0} z. \]

**Standard candles and the Hubble constant**

1. Identify a population of standard candles.
2. Measure redshift \( z \) and flux \( f \) for each standard candle.
3. Compute the luminosity distance \( d_L \) for each standard candle.
4. Plot \( v = cz \) versus \( d_L \), including only candles with \( z \ll 1 \); the slope gives the value of \( H_0 \).

One class of standard candles are Cepheid variable stars with average luminosities in the range 300 \( \to 40,000 \, \text{L}_\odot \) which vary with periods in the range 2 \( \to 60 \, \text{days} \). Brighter stars have longer periods \( \Rightarrow \) we can calibrate our standard candle. With the Hubble Space Telescope, the fluxes and periods of Cepheids can be accurately measured as far \( \sim 20 \, \text{Mpc} \) away \( \Rightarrow H_0 \sim 75 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \).

**Standard candles and the accelerating universe**

In a static, spatially flat universe, \( d_{P,0}, d_L, \) and \( d_A \) would be identical. In an expanding, spatially flat universe

\[ \frac{d_L}{1 + z} = d_{P,0} = d_A(1 + z). \]

For \( z < 1 \),

\[ d_{P,0} \approx \frac{c}{H_0} z \left( 1 - \frac{1 + q_0}{2} z \right), \]

\[ d_L \approx \frac{c}{H_0} z \left( 1 + \frac{1 - q_0}{2} z \right). \]
\[ d_A \approx \frac{c}{H_0} z \left(1 - \frac{3 + q_0}{2} z\right). \]

In the limit \( z \ll 1 \), \( cz \approx H_0 d \) for all distances.

In the limit \( z \to \infty (a_c \to 0, t_c \to 0) \), the current proper distance is equal to the current horizon size:

\[
d_{P0}(z \to \infty) \approx r_H(t_0) = c \int_0^{t_0} \frac{dt}{a(t)},
\]

\[
d_L(z \to \infty) \approx z r_H(t_0),
\]

\[
d_A(z \to \infty) \approx \frac{r_H(t_0)}{z}.
\]

Note that if \( q_0 > -1 \), the value of \( d_A \) has a maximum value at a redshift \( z_c \sim 1 \). An object at very high redshift has a large angular size because the light we detect was emitted when the object was looming large over us.

Plotting \( \delta \theta \) versus \( z \) for a population of standard yardsticks should be a way to determine \( q_0 \). However, a good yardstick is hard to find.

**Type Ia supernovae**

Most attempts to determine the Hubble constant and the deceleration parameter have focused on standard candles. To measure \( q_0 \) your ‘candle’ must be visible at a redshift \( z \sim 1 \) ⇒ Cepheid variable stars are too low in luminosity ⇒ **Type Ia supernovae**.

A supernova is an exploding star. Type Ia supernovae are thought to occur in close binary systems where one member of the binary is a carbon/oxygen white dwarf. Mass is transferred from the companion star to the white dwarf. When the white dwarf exceeds the Chandrasekhar limit of 1.4M_☉, the electron degeneracy pressure can’t support the white dwarf against its own gravity and it collapses until the increased density triggers a runaway nuclear fusion reaction (a fusion bomb!). For a few days, a type Ia supernova in a moderately bright galaxies can outshine all the stars in the galaxy combined.

Type Ia supernovae are not absolutely identical. Fortunately, the peak luminosity correlates with the rise and fall time of the light curve. Type Ia
supernovae that shoot up rapidly and decline rapidly are less luminous than average etc (c.f. Cepheid stars).

In the following, we will make reference to “apparent magnitudes” and “absolute magnitudes”. The **apparent magnitude** is defined as

\[ m \equiv -2.5 \log_{10}(f/f_X) \]

where the reference flux is \( f_X = 2.52 \times 10^{-8} \text{watts m}^{-2} \). The Sun has an apparent magnitude of \( m = -26.8 \) and the stars visible to the naked eye typically have \( 0 < m < 6 \). The **absolute magnitude** is defined as the apparent magnitude that it would have if it were at a luminosity distance of 10 parsecs

\[ M = m - 5 \log_{10}\left(\frac{d_L}{1 \text{Mpc}}\right) + 25 \]

or

\[ M = -2.5 \log_{10}(L/L_X) \]

In the small redshift limit

\[ m \approx M + 43.17 - 5 \log_{10}\left(\frac{H_0}{70 \text{km s}^{-1} \text{Mpc}^{-1}}\right) + 5 \log_{10} z + 1.086(1 - q_0)z \]

Note that \( q_0 \) is related only to the shape of the curve, not its amplitude – even if you totally screw up \( H_0 \) by guessing the wrong absolute magnitude for your standard candles, you can still get the right value of \( q_0 \).

At a given redshift, an **accelerating** universe \((q_0 < 0)\) yields fainter standard candles than a **decelerating** universe \((q_0 > 0)\). Two supernova research groups have seen type Ia supernovae at \( z > 0.5 \) that have a smaller flux than would be expected in decelerating universe \( \Rightarrow q_0 < 0 \).

The predicted differences between \([\Omega_0 = 0.3, \Omega_{\Lambda,0} = 0.7]\) and \([\Omega_0 = 0.3, \Omega_{\Lambda,0} = 0]\) is a few tenths of a magnitude (a difference of \( \sim 25\% \) in the flux) at \( z = 0.5 \). We can now ask: “What values of \( \Omega_{m,0} \) and \( \Omega_{\Lambda,0} \) give the best fit to the observed relation between apparent magnitude and redshift?” Both positively and negatively curved universes are permitted by the supernova data; however, if the universe is assumed to be spatially flat, then \( \Omega_{m,0} \approx 0.3, \Omega_{\Lambda,0} \approx 0.7 \) gives the best fit.
Potential problems: Perhaps the properties of type Ia supernovae were different at $z \sim 0.5$, i.e. not standard candles. Perhaps the type Ia supernovae at $z \sim 0.5$ look faint because of intergalactic dust.

Summary

By measuring distances and redshifts to far-away sources, we can deduce the expansion history of the universe and thus compute the energy densities. We can then extrapolate the expansion into the future to predict the fate of the universe.

Cepheid observations give the expansion rate today (i.e. $H_0$). Type Ia supernovae observations tells us that the universe is currently accelerating and will continue to expand for all eternity. The benchmark model with $\Omega_{m,0} \approx 0.3$, $\Omega_{\Lambda,0} \approx 0.7$ is a good fit to all data.