Lecture 17 Collapse phase I

Last lecture we studied what mechanisms are available to support a cloud and prevent it from collapsing. We will now study in more detail the condition for collapse of an individual core, and what its structure is expected to look like, using some simplifying assumptions.

Bonnor-Ebert spheres

We start off by studying the structure of an isolated core. How does density ρ vary with radial distance, in hydrostatic equilibrium? If we for the moment neglect other forces than gas pressure P (e.g. no rotation or $\vec{B}$), we recall the equation of hydrostatic equilibrium

$$\boxed{-\frac{1}{\rho}\nabla P - \nabla \phi = 0}\,,$$ where ϕ is the

gravitational potential. Let us furthermore assume an isothermal eq. of state $P = c_s^2 \rho$, where c_s is the sound speed. Together, these relations imply that $\ln \rho + \frac{\Phi}{c_s^2} = \text{constant}$, as we see by applying the gradient:

$$\bar{\nabla}\left(\ln\rho + \frac{\Phi}{c_s^2}\right) = \frac{1}{\rho}\bar{\nabla}\rho + \frac{\nabla\Phi}{c_s^2} = \left\{ \bar{\nabla}\rho = \frac{\bar{\nabla}P}{c_s^2} \right\}$$

$$= \frac{1}{c_s^2}\left(\frac{1}{\rho}\bar{\nabla}P + \bar{\nabla}\Phi\right) = 0$$

$$\Rightarrow \underline{-\frac{1}{\rho}\bar{\nabla}P - \bar{\nabla}\Phi = 0}$$

(which is the hydrostatic equilibrium equation). We now limit ourselves to a spherical symmetric core, and define a density structure fullfilling hydrostatic equilibrium by rewriting $\ln\rho + \frac{\Phi}{c_s^2} = \text{const.}$ to $\rho(r) = \rho_c \exp\left(-\frac{\Phi}{c_s^2}\right)$.

But what is Φ? Φ is related to ρ via the Poisson equation

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$\nabla^2 \Phi = 4\pi \overset{\downarrow \text{gravitational const.}}{G} \rho$, which in spherical

coordinates is $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \rho$

for spherically symmetric geometries. We thus have the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \underbrace{\rho_c \exp\left(-\frac{\Phi}{c_s^2}\right)}_{=\rho \text{ as we defined it}}.$$

To get rid of all the constants we define new variables $\xi = \underbrace{\sqrt{\frac{4\pi G \rho_c}{c_s^2}}}_{\text{const.}}\, r$ and $\psi = \frac{\Phi}{c_s^2}$,

which gives us (verify)

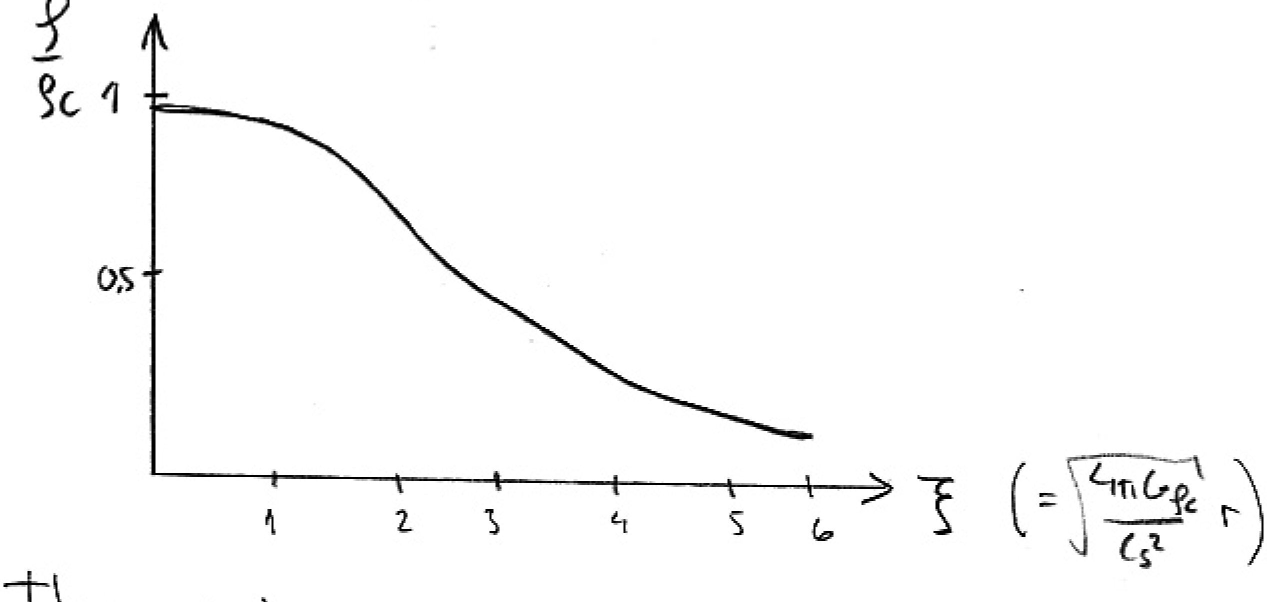
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = \exp(-\psi).$$

$= \frac{\rho}{\rho_c}$

This equation is called the <u>isothermal Lane-Emden equation</u>, from which we can derive $\rho(r)$ as a function of ρ_c. The physical boundary conditions we impose on the equation is that 1) $r \to 0 \Rightarrow \rho \to \rho_c$, or equivalently, $\xi \to 0 \Rightarrow \psi \to 0$.

2) From spherical symmetry, the force $F \to 0$ as $r \to 0$, or equivalently $\xi \to 0 \Rightarrow \frac{d\psi}{d\xi} \to 0$.

These two boundary conditions gives us a unique solution to the differential equation, which, however, needs to be solved numerically. The solution (expressed as $\frac{\rho}{\rho_c}(\xi)$) looks something like this:



The solution is thus essentially parametrised by ρ_c (and c_s), as is the total mass, which diverges for $\xi \to \infty$. More realistic are models that do not extend to ∞, but are limited by the external pressure P_{ext} from the parent cloud on ISM. In that case, the core only extends out to the radius ξ_{ext} such that $P(\xi_{ext}) = c_s^2 \rho(\xi_{ext}) = P_{ext}$. The total mass of the sphere then becomes (integrating over spherical shells)

$$M = \int_0^{r_{ext}} \rho \, 4\pi r^2 dr = 4\pi \rho_c \left(\frac{c_s^2}{4\pi G \rho_c} \right)^{\frac{3}{2}} \int_0^{\xi_{ext}} e^{-\psi} \xi^2 d\xi =$$

$$= \left\{ \frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi} \xi^2 \right\} = 4\pi \rho_c \left(\frac{c_s^2}{4\pi G \rho_c} \right)^{3/2} \left[\xi^2 \frac{d\psi}{d\xi} \right]_0^{\xi_{ext}}$$

To clean up constants, we again introduce a scaled quantity, this time a dimensionless mass $m \equiv \dfrac{M}{\left(\dfrac{c_s^2}{\rho_{ext}^{1/3} G} \right)}$. Using that $\left.\dfrac{d\psi}{d\xi}\right|_0 = 0$,

$$\underline{m = \sqrt{\frac{\rho_{ext}}{4\pi \rho_c}} \left(\xi^2 \frac{d\psi}{d\xi} \right) \Big|_{\xi = \xi_{ext}}}$$

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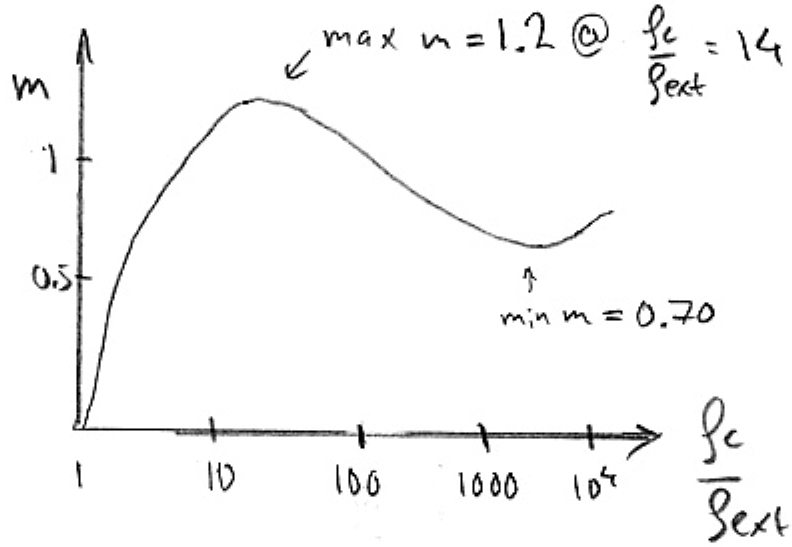
The dimensionless mass m exhibits an interesting dependency on the ρ_{ext}/ρ_c ratio, as qualitatively shown in the following figure:



As seen in the figure, there is a maximum mass $m_{max} \approx 1.2$, with no solutions for $m > m_{max}$. m_{max}, or its dimensional corresponding quantity $M_{BE} = \frac{m_{max} C_s^4}{\sqrt{\rho_{ext} G^3}}$ is called the Bonnor-Ebert mass. Since there are no static solutions of $M > M_{BE}$, we must have either collapse, explosion, or oscillation. It can be shown that for $\frac{\rho_c}{\rho_{ext}}$ to the left of the peak, there are stable oscillations, while to the right collapse is inevitable. What happens then? To study the collapse, introduce M_r, denoting the mass within radius r:

$$M_r \equiv \int_0^r 4\pi r^2 \rho \, dr \Rightarrow \boxed{\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho}$$

Furthermore, $\boxed{\frac{\partial M_r}{\partial t}} = \int_0^r 4\pi r^2 \frac{\partial \rho}{\partial t} \, dr =$

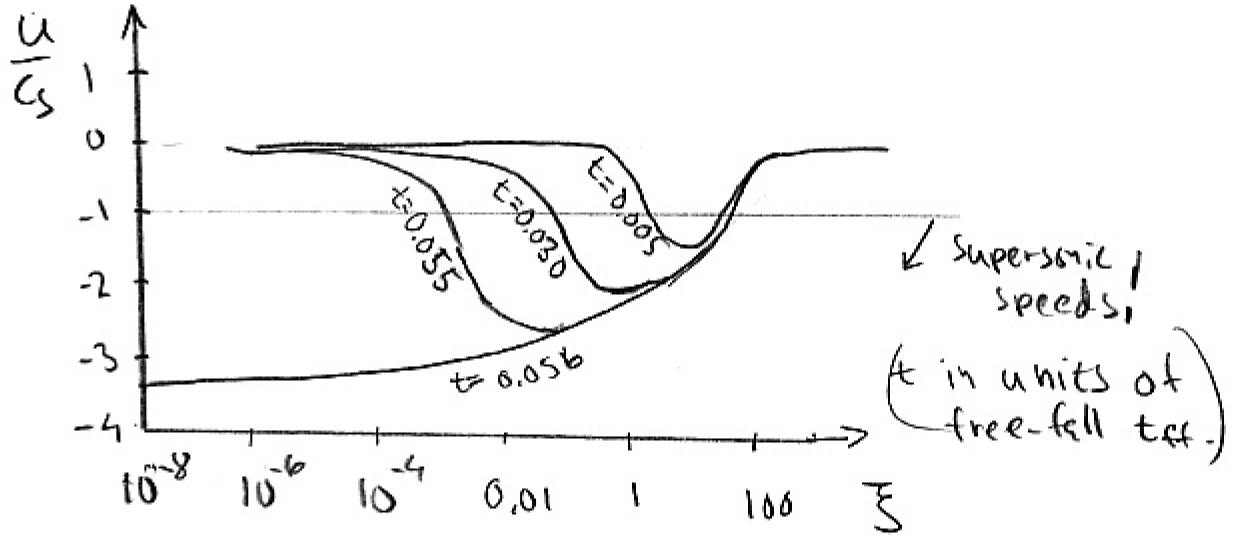
$$= \left\{ \frac{\partial \rho}{\partial t} = -\frac{1}{r^2} \frac{\partial (r^2 \rho u)}{\partial r} \text{ (continuity eq.)} \right\} =$$

$$= -4\pi \left[r^2 \rho u \right]_{r=0}^{r} = \boxed{-4\pi r^2 \rho u.}$$

Since in spherically symmetry, Newton showed that $F = -\frac{GM_r}{r^2} m$, we have the momentum equation

$$\boxed{\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r} - \frac{GM_r}{r^2}}$$

Together, these three coupled differential equations can be numerically integrated to get the time-dependence of the collapse. As a boundary condition, one often assumes constant external pressure. It is thus assumed that the external cloud compensates for the collapse. An example of a solution is given in Fig. 10.5 of S&P:



The collapse starts in the outer edge and expands inwards. The whole mass falls into the centre in a time on the order of a few free-fall time-scales

The Bonnor-Ebert sphere solutions are interesting because they give an idea of the radial structure of dense cores, and how they respond to external pressure, BUT there are many shortcomings:

* As we previously discussed, gas pressure and gravity are not the only relevant forces; magnetic fields and turbulence are likely very important.
* The simplified equation of state introduced by the assumption of isothermality ignores important heating / cooling balance that might introduce inhomogeneities in the thermal structure. In particular, gas that cools loses pressure.
* Cloud cores are not spherically symmetric static structures, but dynamic and strongly heterogeneous.

Star formation is a violently dynamic process producing groups of stars at the same time, with a high fraction being <u>multiple</u> stars. 17:10

Rotating cores

One obvious extension to our collapsing models is to introduce rotation. How does it change the core structure? Since we no longer have spherical symmetry, we introduce cylindrical coordinates instead, with r being the cylindrical radius, φ the azimuthal coordinate, z being parallell to the rotational axis. Call the specific angular momentum $j \equiv r\, u_\varphi$ (azimuthal velocity), then the inertial centrifugal force can be introduced as $\frac{j^2}{r^3}$ into the hydrostatic equilibrium equation:

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial P}{\partial r} - \dfrac{\partial \Phi}{\partial r} = -\dfrac{j^2}{r^3} & \text{(in rotational plane)} \\ -\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} - \dfrac{\partial \Phi}{\partial z} = 0 & \text{(along rotational axis)} \end{cases}$$

Using an isothermal e.o.s. $P = c_s^2 \rho$,

$$\begin{cases} -\dfrac{c_s^2}{\rho}\dfrac{\partial \rho}{\partial r} - \dfrac{\partial \Phi}{\partial r} = -\dfrac{j^2}{r^3} \\[2ex] -\dfrac{c_s^2}{\rho}\dfrac{\partial \rho}{\partial z} - \dfrac{\partial \Phi}{\partial z} = 0 \end{cases}$$

A particular result is that j only depends on r, not z. To see this, derive the first equation by z and the second by r, and subtract:

$$-\frac{\partial}{\partial z}\left(\frac{j^2}{r^3}\right) = \frac{\partial}{\partial z}\left(-\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{\partial \Phi}{\partial r}\right) - \frac{\partial}{\partial r}\left(-\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial z} - \frac{\partial \Phi}{\partial z}\right)$$

cancel

$$\Leftrightarrow \quad -\frac{1}{r^3}\frac{\partial j^2}{\partial z} = -c_s^2 \underbrace{\left(\frac{-1}{\rho^2}\frac{\partial \rho}{\partial z}\frac{\partial \rho}{\partial r} \;\; \frac{1}{\rho}\frac{\partial^2 \rho}{\partial r \partial z} - \frac{-1}{\rho^2}\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z} - \frac{1}{\rho}\frac{\partial^2 \rho}{\partial z \partial r}\right)}_{=0}$$

so $\dfrac{\partial j^2}{\partial z} = 0$, and j is independent of z.

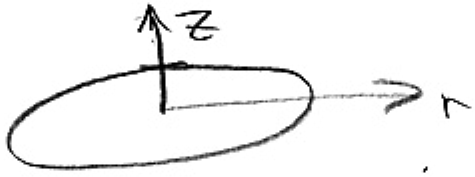
$\Rightarrow$ we can define a centrifugal potential

$$\Phi_{cen} \equiv -\int_0^\infty \frac{j^2}{r^2}\,dr$$

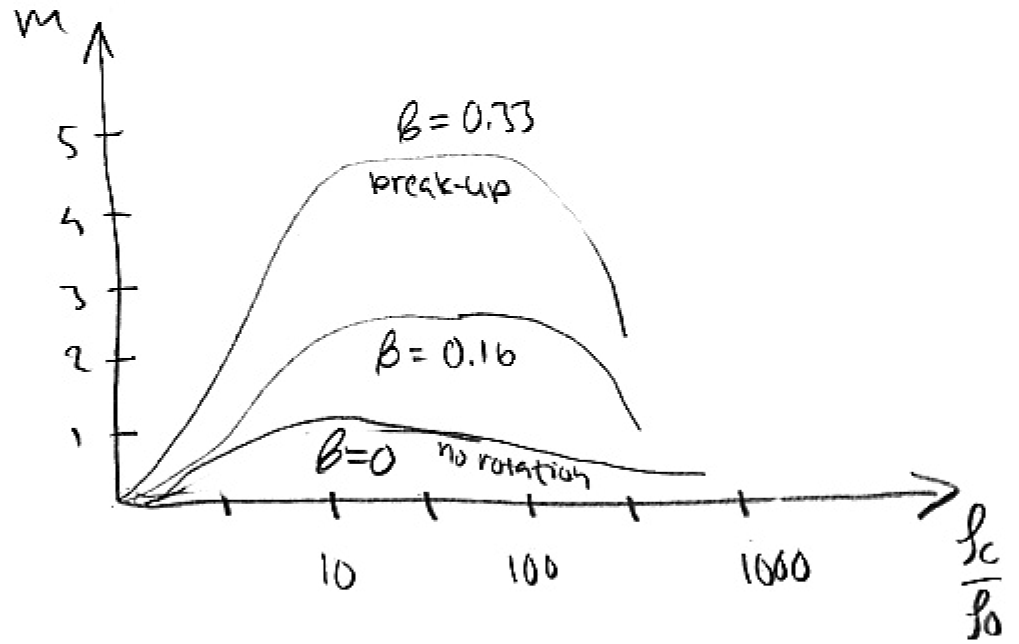
and write the hydrostatic equilibrium equation as

$$\underbrace{-\frac{c_s^2}{\rho}\bar{\nabla}\rho}_{\text{isothermal pressure}} - \underbrace{\bar{\nabla}\Phi_g}_{\text{gravity potential}} - \underbrace{\bar{\nabla}\Phi_{cen}}_{\text{centrifugal potential}} = 0.$$

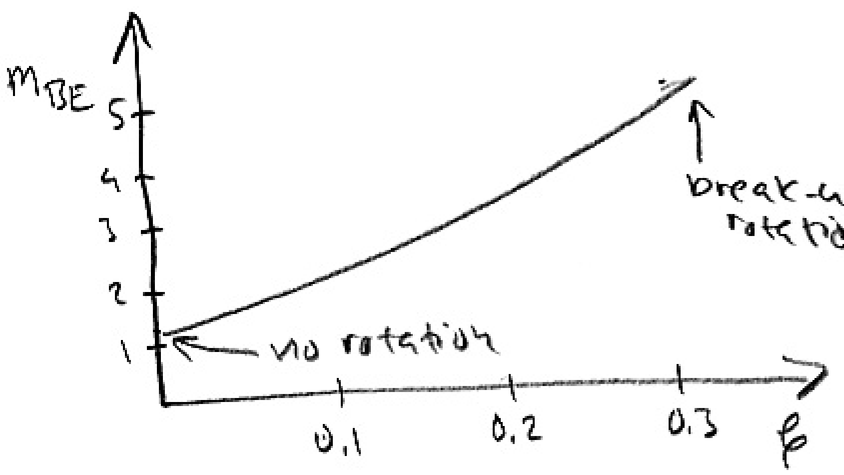
The resulting structure is (as expected) flattened to an oblate spheroid:



To study how rotation affects the Bonnor-Ebert mass, it is common to introduce a rotational parameter $\beta = \frac{\Omega_0^2 R_0^3}{3GM}$, where Ω_0 and R_0 is the initial angular velocity and radius of the collapsing core. $\beta = \frac{1}{3}$ corresponds to break-up speed.



The rotation thus increases the M_{BE} significantly as a function of β

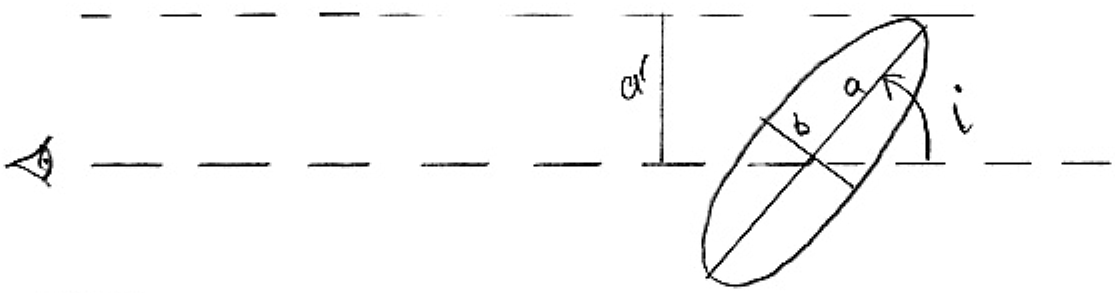


Observations show, however, that rotation is <u>not</u> responsible for the cloud core elongations (the observed rotational velocities are far too low). Moreover, rotating cloud cores would be <u>oblate</u>, while observed cloud cores are more consistent with being statistically <u>prolate</u>. (oblate is discuss-shaped while prolate is more like a cigar)

<u>Statistics of cloud core shapes</u>

To see how observations favour prolate over oblate, we study the relation between the observed aspect ratio $R_{app} = \frac{\text{minor axis}}{\text{major axis}}$

as a function of projection $\cos i$ for the case of prolate and oblate spheroids.



In S&P it is shown that

$$\sin^2 i = \frac{1 - R_{app}^2}{1 - R_{true}^2} \quad \text{for oblate}$$

and

$$\cos^2 i = \frac{R_{true}^{-2} - R_{app}^{-2}}{R_{true}^{-2} - 1}.$$

This can be recast to

$$R_{app}^{oblate}(\mu) = \sqrt{1 - (1-\mu^2)(1-R_{true}^2)}$$

and

$$R_{app}^{prolate}(\mu) = \sqrt{R_{true}^{-2} - \mu^2(R_{true}^{-2} - 1)},$$

where $\mu = \cos i$. In two figures on the next page are these relations plotted for 9 ratios between 0.1 and 0.9 ($R_{true} = 1.0$ is a sphere, and so the projection will be a circular disc independent of inclination).

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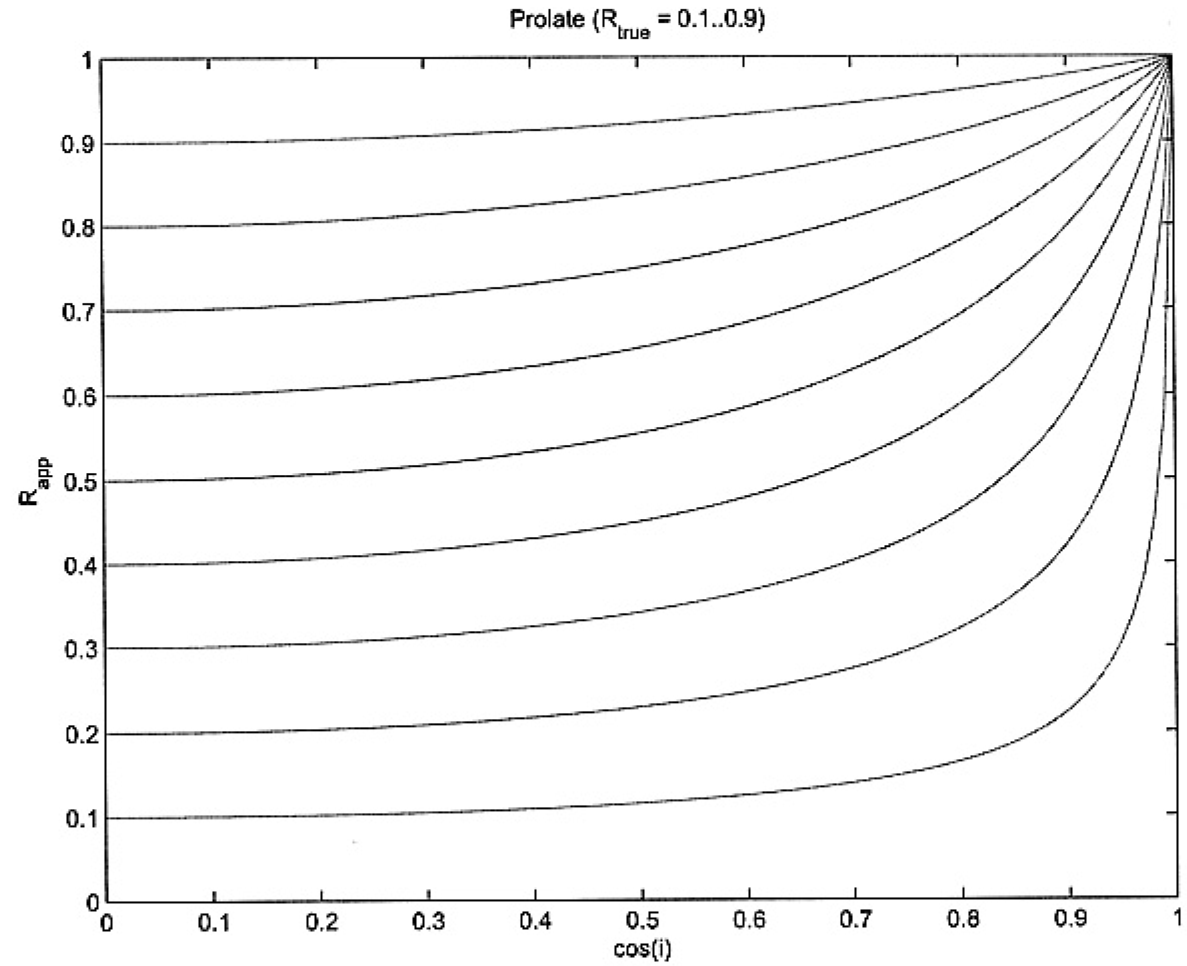
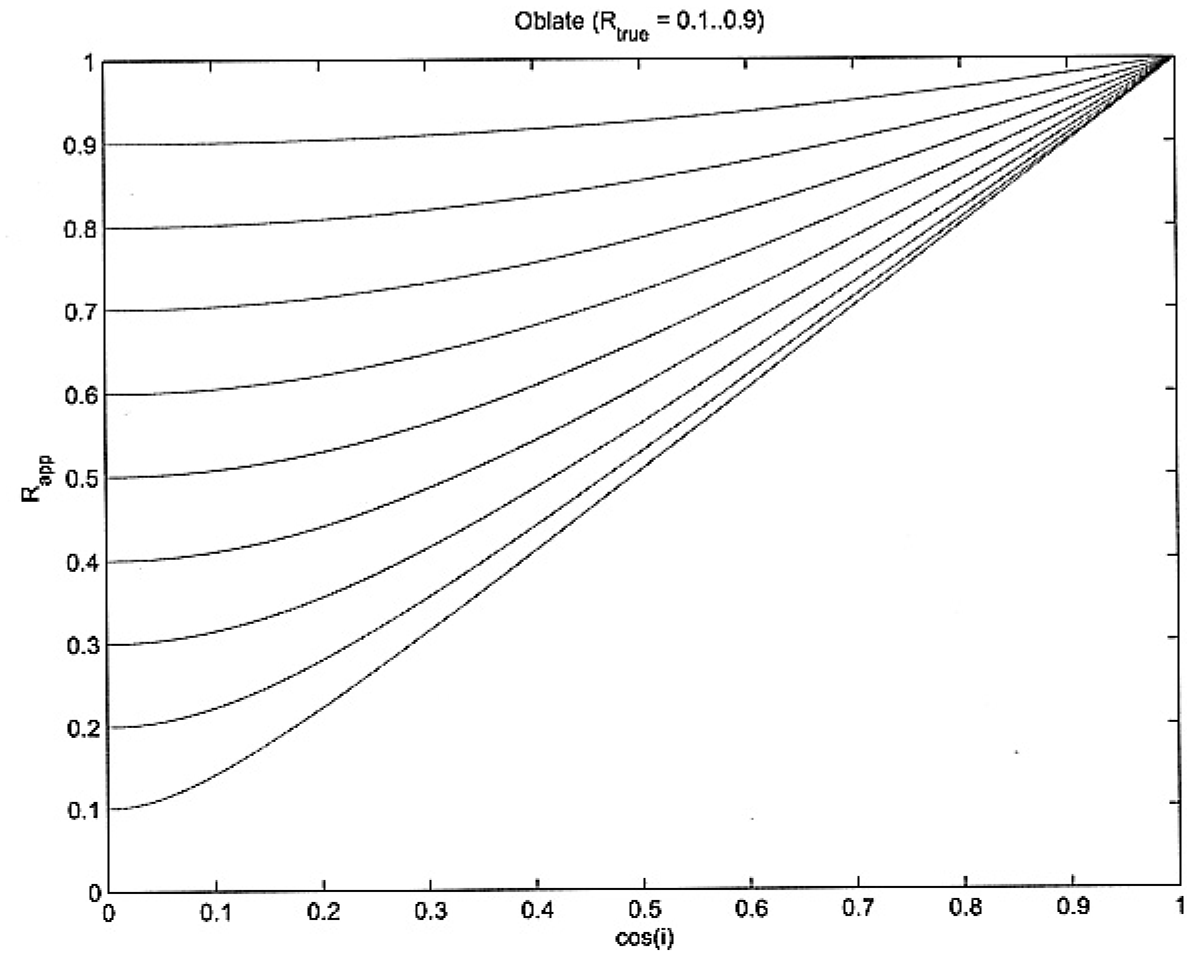

Prolate (R_{true} = 0.1..0.9)
R_{app}
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
cos(i)


Oblate (R_{true} = 0.1..0.9)
R_{app}
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
cos(i)

Note the difference: to produce a small apparent ratio is impossible for an oblate disk at high inclinations, however thin.
The observed apparent axis ratio of dense cores is about 0.6. Since $\cos i$ is uniformly distributed between 0 and 1, $\langle \cos i \rangle = \frac{1}{2}$, and the expectation value for R_{app} is $\langle R_{app} \rangle = \int_0^1 \mu R_{app}(R_{true}, \mu)\, d\mu$, where R_{true} is assumed to be given.
Prolate spheroids can reproduce any $\langle R_{app} \rangle$, but the smallest $\langle R_{app} \rangle$ an infinetelly thin oblate spheroid can produce is $\langle R_{app} \rangle = 0.5$.
So, to reproduse the observed $\langle R_{app} \rangle \approx 0.6$ we have to have a distribution of either extremely thin oblate spheroids, or modestly elongated prolate spheroids. Since extremely thin spheroids are hard to justify theoretically (how are they maintained so thin?), prolate spheroids are thought to be favoured.