PART A

Stellar evolution foundation

Welcome to the ball game!

A star can, in a simplest framework, be defined by

- **Birth mass** ***M***
- **Metallicity** ***Z.*** Reduction to a single parameter possible as many stellar generations and mixing processes damp out fluctuations in individual galactic regions. X + Y + Z = 1, where X is H mass fraction, Y is He mass fraction and Z is the mass fraction of all other elements.
- **Angular momentum ("rotation")** ***Ω.*** Reduction to a single parameter possible because early MS brings about close to rigid-body rotation independent of the initial profile. The star can later develop a differential profile.

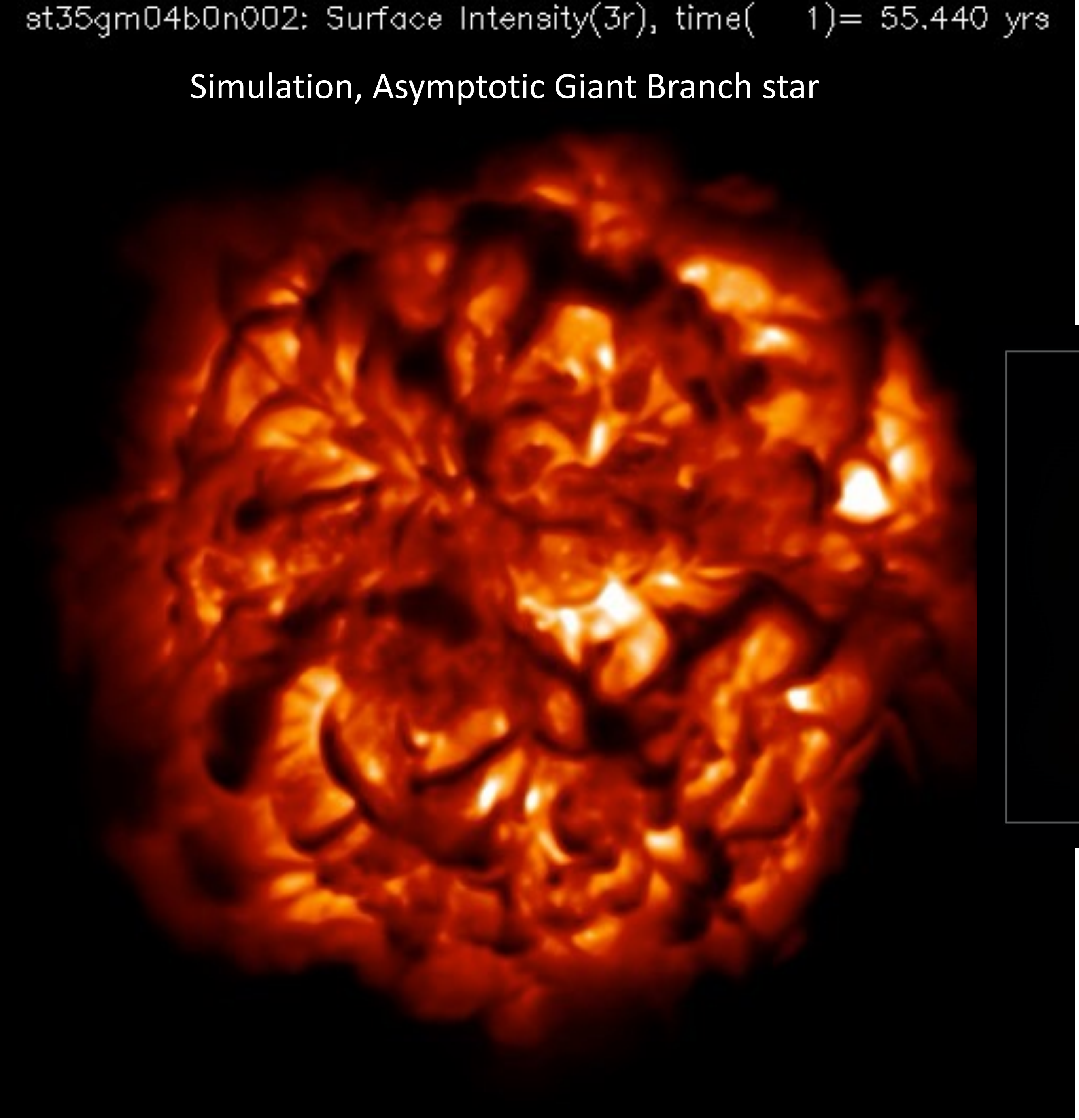
But also:

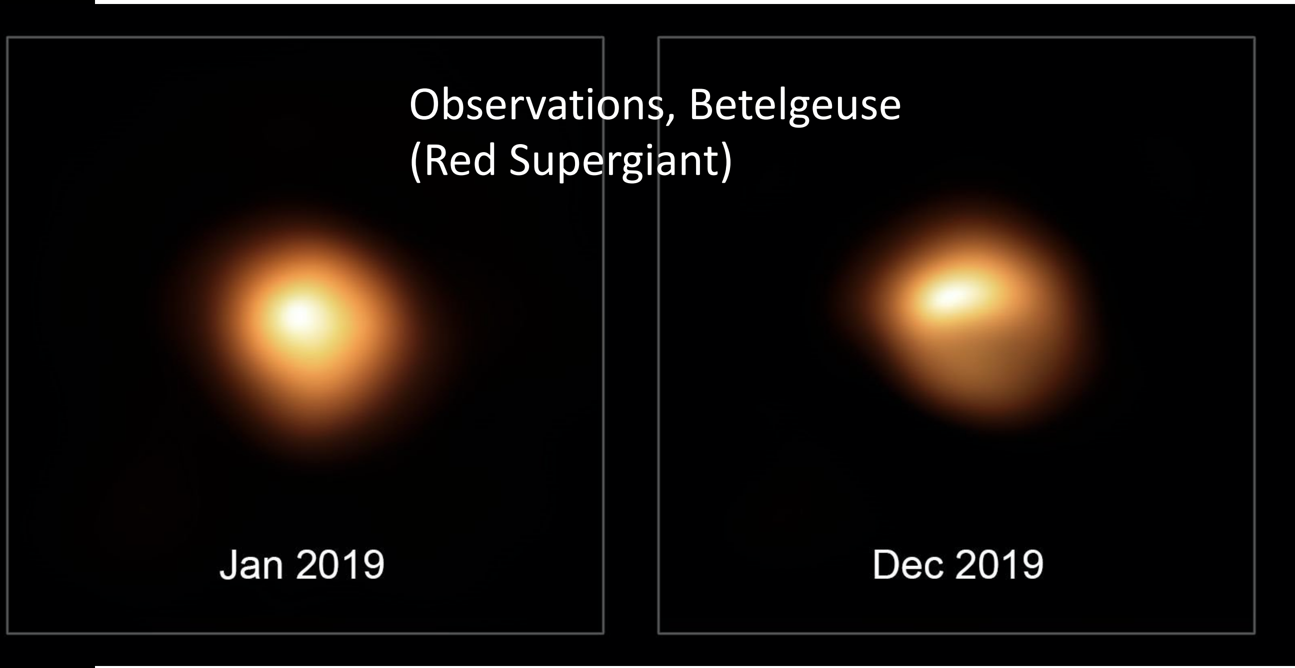
- **Magnetic field strength** ***B ?***
- **Binary companion (*M_2*, initial separation *d*, *Ω_2*)**: removes or adds mass (which can also change the angular momentum) at certain points in time.

And perhaps:

- **Initial conditions?** A philosophical question: *To what extent do small random differences in initial conditions lead to a fundamentally different star later on?* The question has obtained recent new impetus for supernova studies where it has become clear that very small perturbations in later burning stages can give very different outcomes for the collapsing star.

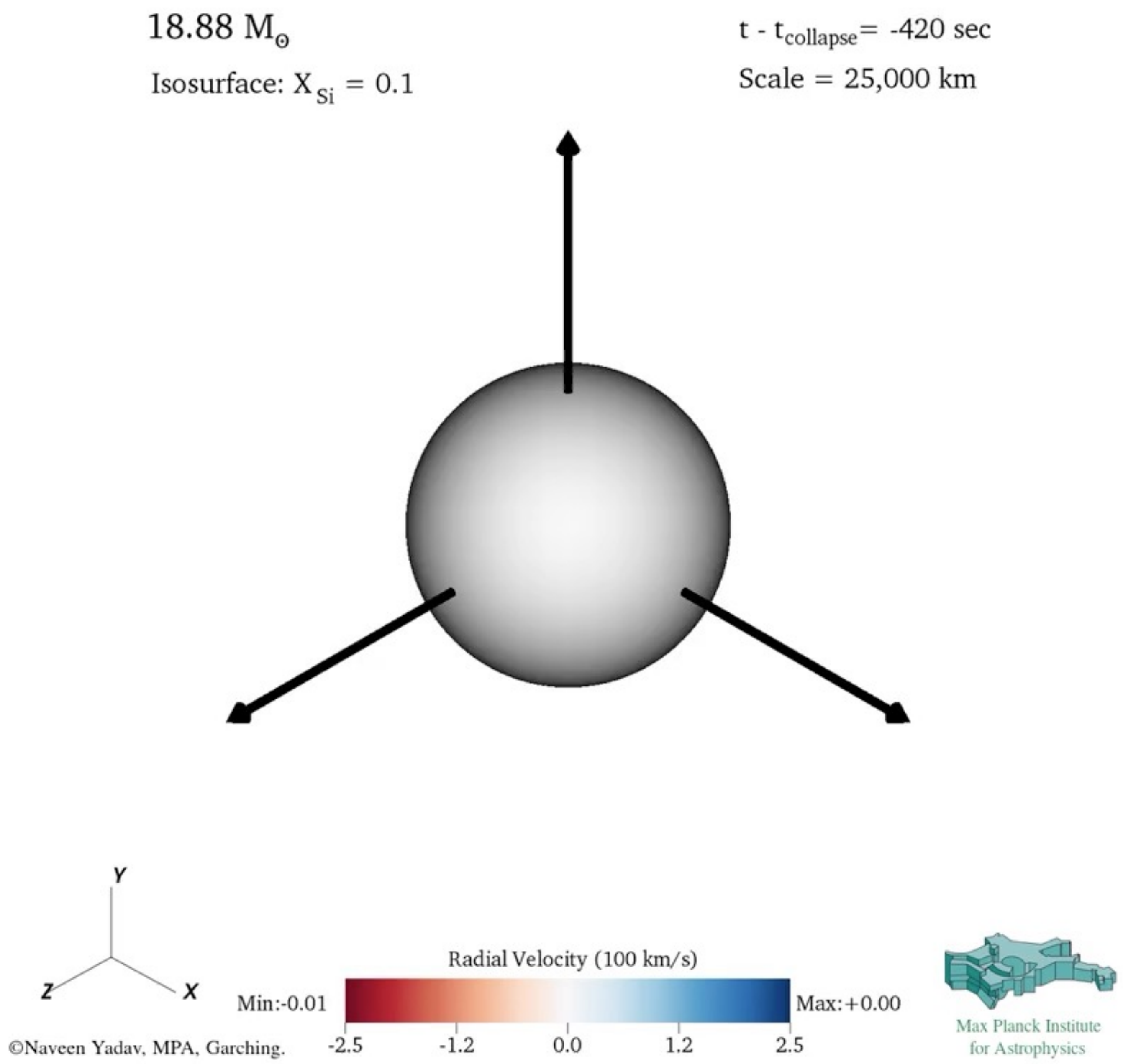
For most their lives stars behave pretty well. But in late stages…





ESO/Montargès et al. 2020

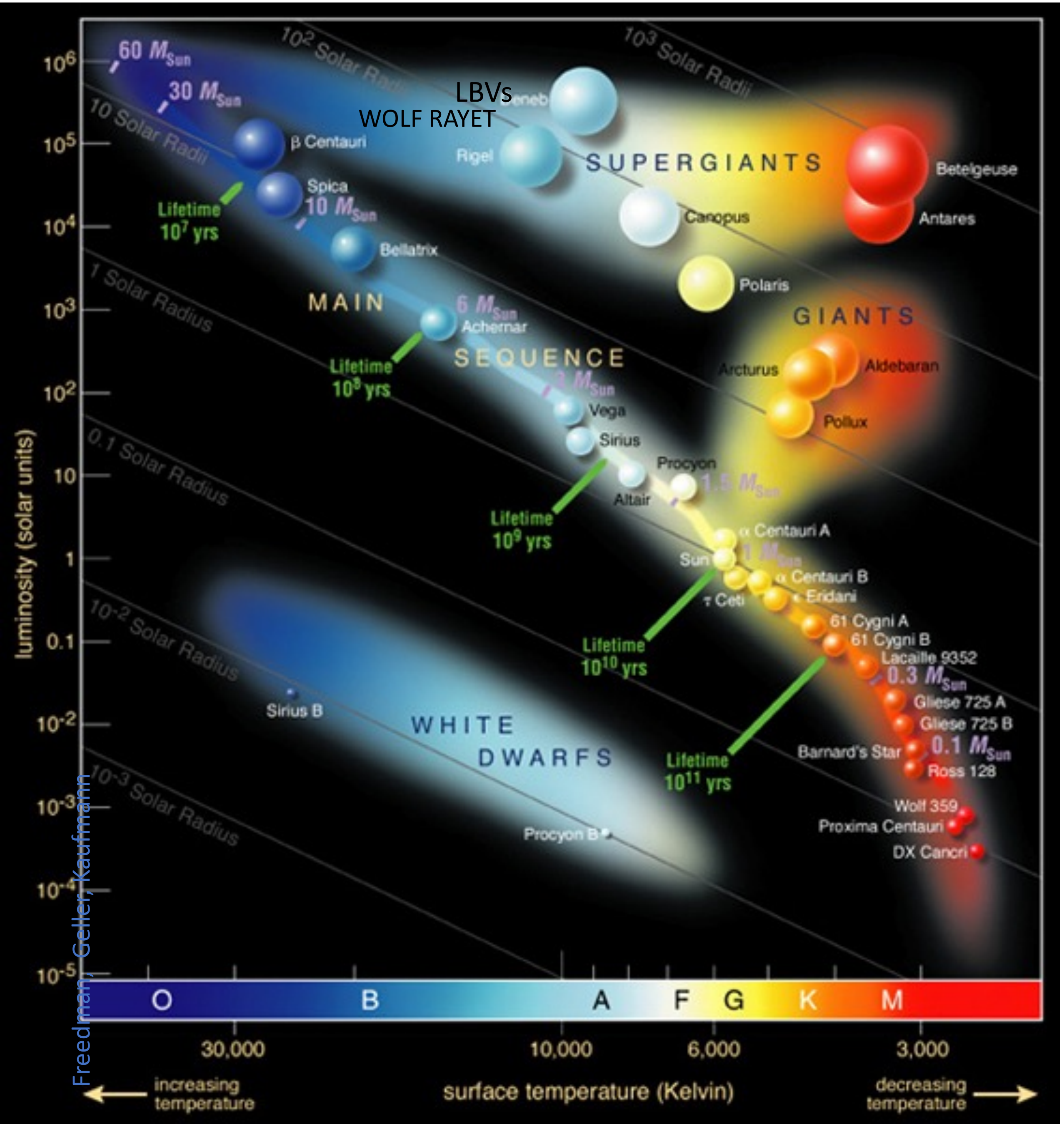
[Freitag, Liljegren, Höfner 2017]



The final 420 seconds in the core of a 18.88 M_{sun} star : the O/Si/S shell merges with the O/Ne/Mg shell.

https://wwwmpa.mpa-garching.mpg.de/ccsnarchive/movies/Yadav_2019/si_iso_vel_pseudocol_smooth.mp4 Yadav+2020

The HR diagram



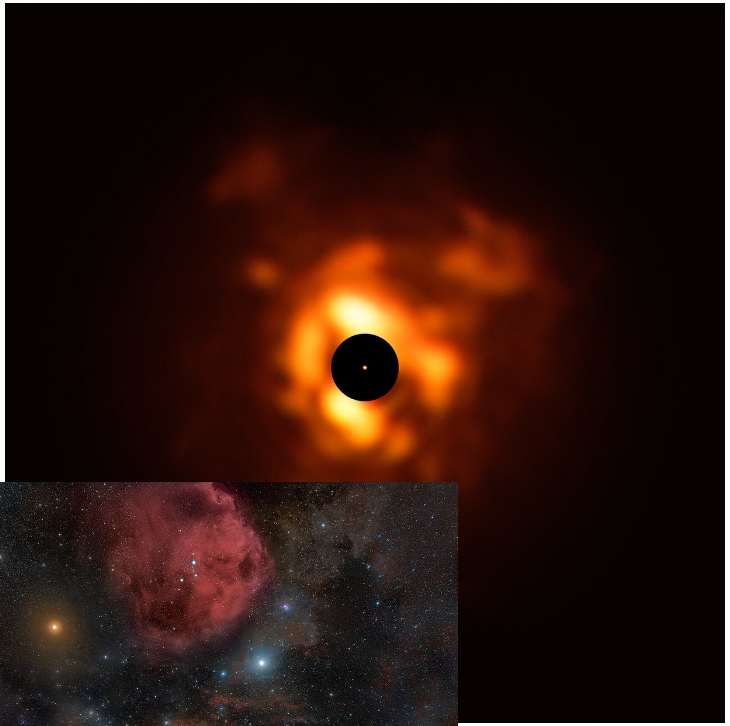
- Luminosity and surface temperature on the y and x axes (the two things you can most easily measure).
- **A main sequence band** (~90% of stars) and **post-MS islands** (~10%) – white dwarfs, giants, red supergiants, yellow supergiants (e.g. Polaris), blue supergiants, Luminous Blue Variables, Wolf-Rayet stars.

On the **main sequence**:

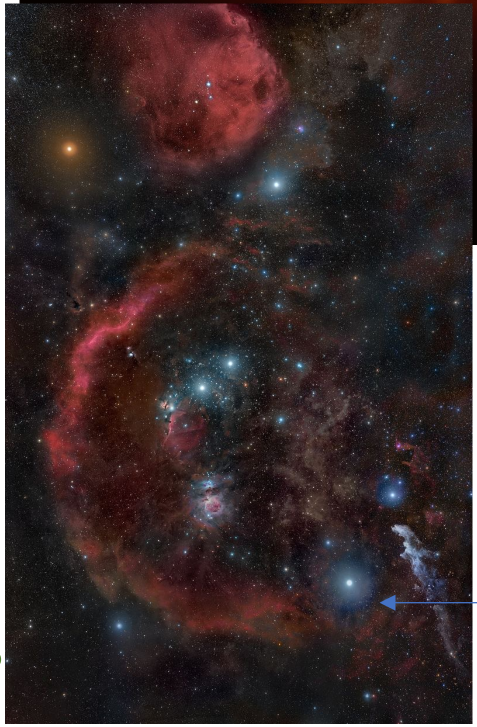
- $L \sim T^a$ with some exponent *a.*
- Since $L \sim R^2 T^4$ for a black-body, R must scale weaker with T than T^{-2} for L to increase with T as observed.
- Smallest stellar masses ~ 0.08 $M_\odot$ at ~10^{-4} L_{sun}: below this mass central temperature never reaches the H ignition temperature.
- Largest stellar masses ~100-200 $M_\odot$ at ~10^6 L_{sun} : regulated by Eddington limit (or possibly tighter limit by star formation constraints).
- Stars vary factor $\sim 10^3$ in mass but factor $\sim 10^{10}$ in luminosity.
- There is little scatter → a single dominant parameter governs both L and T. Is it mass?

A preview of late stellar evolution: some famous post-MS stars

Red supergiant **Betelgeuse**



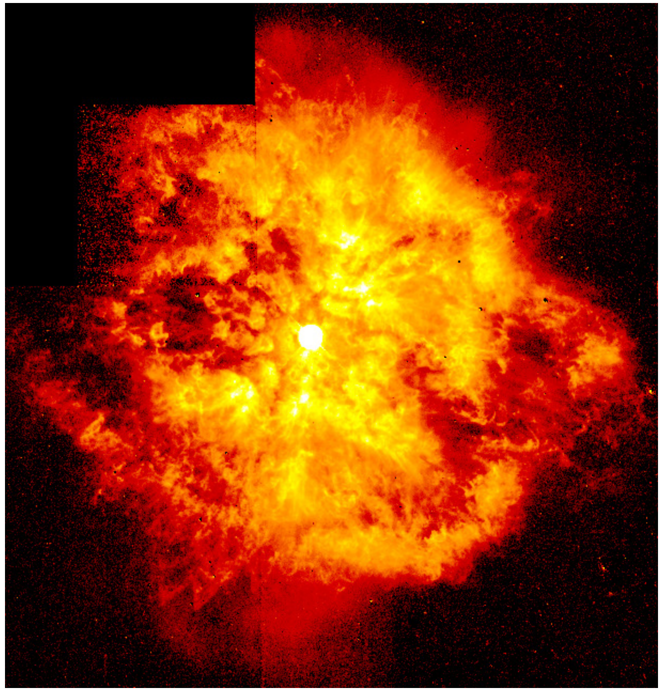
ESO/P. Kervella



Rogelio Bernal Andreo

Blue supergiant **Rigel**

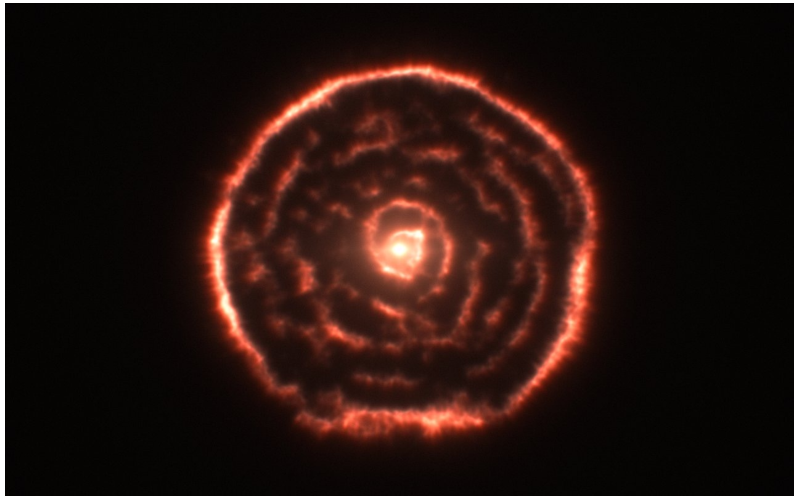
Wolf-Rayet star **WR 124**



NASA/STScI

Luminous Blue Variable **Eta Carina**

NASA/Jon Morse



ESO/M.Maercker

Red Giant **R Sculptoris**

From spatially resolved cases: clearly complex stars with strong mass loss.

The initial mass function

- About 90% of stars are $M_{ZAMS} < 0.4\ M_{\odot}$ red dwarfs: will reach post-MS stage only in time >> age of the universe.

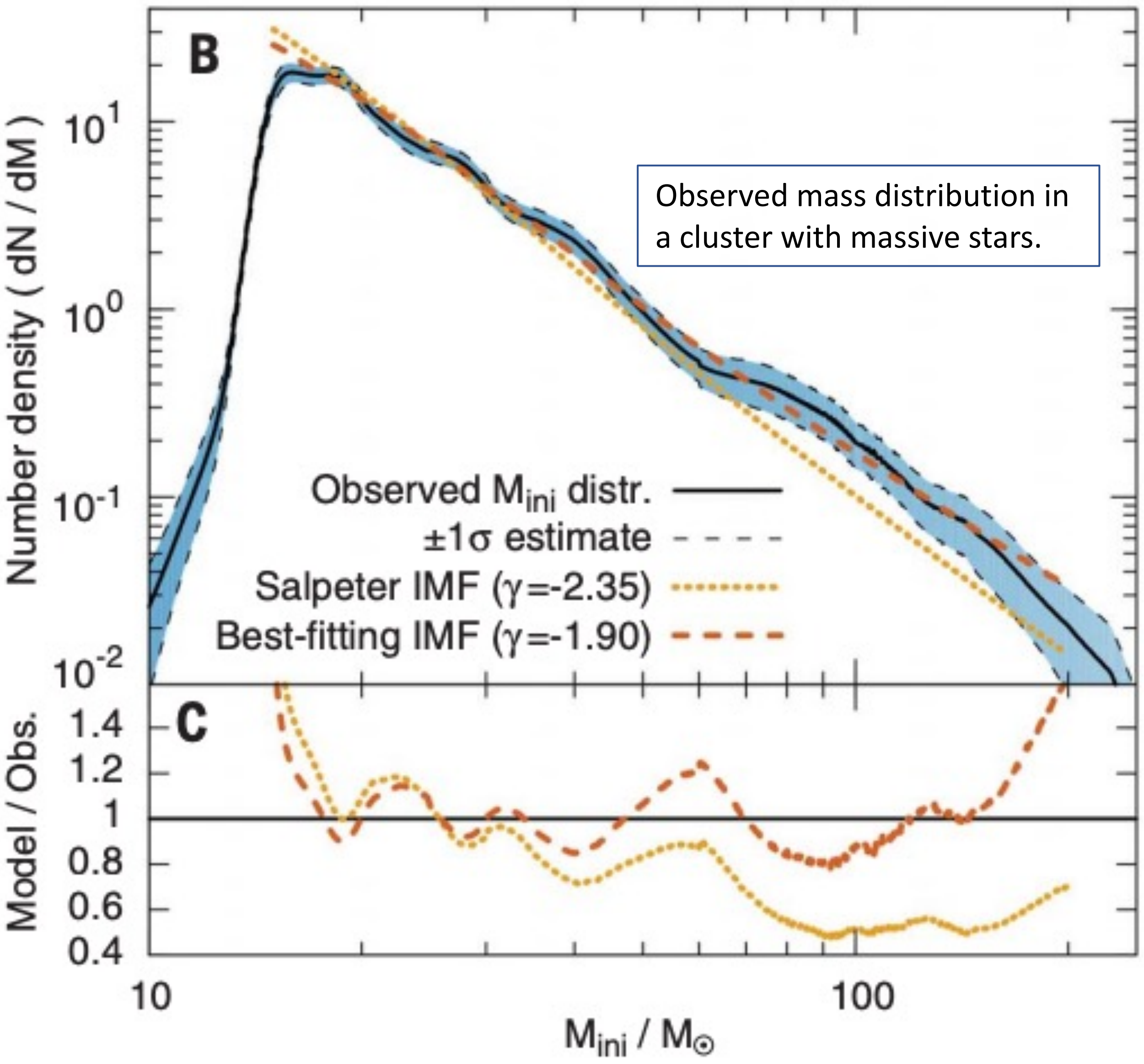
- Distribution of masses at birth well described by the **Salpeter distribution**,

$$\frac{dN}{dM_{ZAMS}} \sim M_{ZAMS}{}^{-2.35},$$

over the whole mass range 0.08 – 200 $M_{\odot}$.

Few stars will become core-collapse supernovae (**$M_{ZAMS} \gtrsim 10\ M_{\odot}$ needed**, as we will se later), and of supernovae the vast majority come from 10-20 $M_{\odot}$ stars:

- One 10 $M_{\odot}$ star for 223 1 $M_{\odot}$ stars.
- One 40 $M_{\odot}$ star for 26 10 $M_{\odot}$ stars
- One 200 $M_{\odot}$ star for 1100 10 $M_{\odot}$ stars

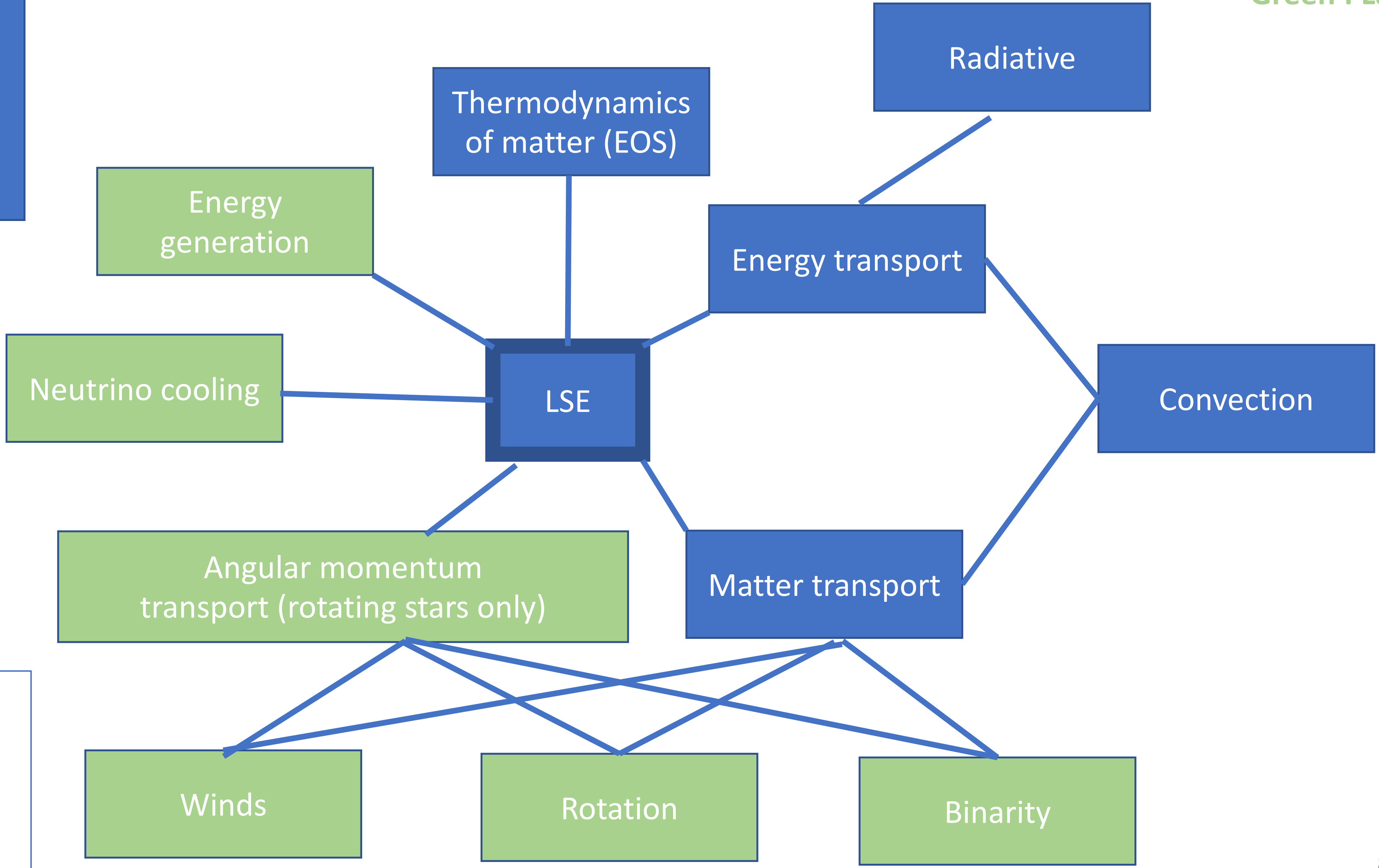


Schneider+2018,Science

Components of late stellar evolution

Blue : Part A
Green : Later

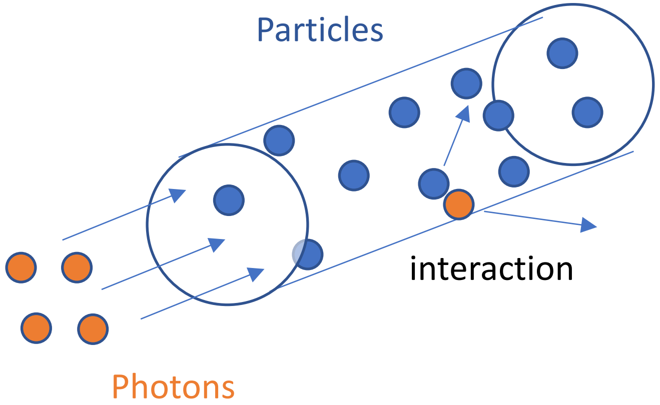
Some fundamental microphysics and timescales (into all boxes)



Plus hydrostatic equilibrium,
mass conservation,
energy conservation.

Fundamental microphysics and time-scales

Photon-matter interaction



Cross section:

$$\sigma = \frac{\text{Number of hits per unit time per target } [\mathrm{s}^{-1}]}{\text{Flux of particles } [\mathrm{cm}^{-2}\mathrm{s}^{-1}]}$$

E.g. electron scattering $\sigma_{Th} = 6.65 * 10^{-25}\ \mathrm{cm}^2$ "Thomson cross section"
In general case cross sections are frequency-dependent.

Mean-free path:

$\lambda_{mfp} \equiv \frac{1}{\sigma n_{targets}}$ [cm] "Mean distance travelled between interactions"

Opacity:

$\kappa \equiv \frac{1}{\rho \lambda_{mfp}} = \frac{\sigma n_{targets}}{\rho}$ [cm^2 g^{-1}] "Cross section per unit weight of material".

Note if gas is completely ionized $n_{targets}/\rho$ has no (ρ, T) dependency : completely specified by the composition.

Fully ionized H gas: $\kappa_{Th} = \frac{\sigma_{Th}}{m_p} = \mathbf{0.4\ cm^2\ g^{-1}}$. Solar H/He mixture: 0.34 cm^2 g^{-1}.

Rosseland opacity

Rosseland opacity: The particular frequency-integral of κ_ν

$$\kappa_R = \frac{acT^3}{\pi}\left(\int_0^\infty \frac{1}{\kappa_\upsilon}\frac{\delta B}{\delta T}\,d\nu\right)^{-1}$$

that allows recovery of the correct bolometric flux by:

$$F = \frac{4acT^3}{3\kappa\rho}\,\kappa_R^{-1}\nabla T$$

Mean molecular weights

$$P = nkT = kT \sum \frac{\rho x_i * (1 + y_i)}{\mu_i} = kT\rho \sum \frac{x_i * (1 + y_i)}{\mu_i} = kT \frac{\rho}{\mu}$$

x_i mass fraction of element *i*
y_i ionization degree of element *i*

μ_i: Molecular weight of species *i*. A molecular weight unit is the "amu" which is approximately the mass of a proton/neutron (1/12 of a ^{12}C nucleus mass, exactly). $\mu_i \approx A_i$ (atomic weight).

Definition $\mu \equiv \left(\sum \frac{x_i*(1+y_i)}{\mu_i}\right)^{-1}$ = "**Mean molecular weight per particle**" (counting all atoms, ions and free electrons).

Similarly, define

$\mu_e \equiv \left(\sum \frac{x_i*y_i}{\mu_i}\right)^{-1}$ ="**Mean molecular weight per free electron**"

and

$\mu_{ion} \equiv \left(\sum \frac{x_i}{\mu_i}\right)^{-1}$ = "**Mean molecular weight per ion**"

(includes also neutral atoms)

Species	μ, fully ionized	μ_e, fully ionized	μ, neutral	μ_e, neutral
H	0.5	1	1	∞
He	1.33	2	4	∞
C	1.71	2	12	∞
Z-> ∞	2	2	∞	∞

Radiative diffusion

How long does it take for photons to travel from center to surface of the star? Random walk of photons:

Mean distance travelled after *N* steps

N = number of steps

Mean-free path

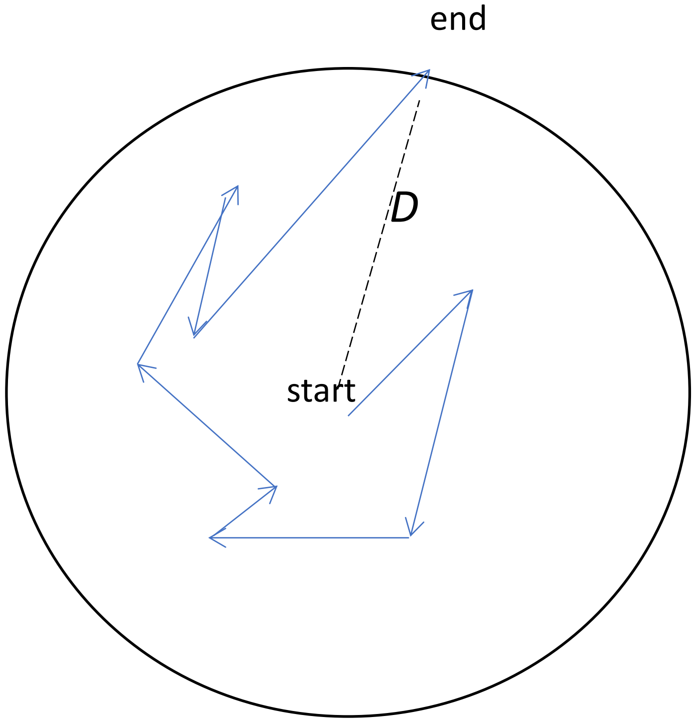
$$\langle D\rangle = \frac{1}{3}N^{1/2}\lambda_{mfp}$$

$$t_{diff} = N\frac{\lambda_{mfp}}{c} = \left(\frac{3\langle D\rangle}{\lambda_{mfp}}\right)^2\frac{\lambda_{mfp}}{c} = \frac{9\langle D\rangle^2}{c\lambda_{mfp}} = \frac{9\langle D\rangle^2\kappa\rho}{c}$$

$$\rho \approx \frac{M}{\frac{4\pi}{3}R^3}$$

$$\rightarrow\ t_{diff} = \frac{27\kappa M}{4\pi c R} = 26{,}000y\left(\frac{\kappa}{0.4\ \mathrm{cm}^2/\mathrm{g}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}$$

Put $\langle D\rangle = R$ to get time to reach surface of star



Specific heat capacities c_p and c_V

Specific heat capacity : how much heat needs to be added to a substance, per unit mass, to increase the temperature by 1 K? (unit erg/g/K)

What is "**heat**"? It is (expressed per unit mass) $dq = du + pdV$, where du is change in internal energy (per unit mass) and dV is the change in specific volume.

The process comes in two flavors:

1. under **constant pressure** (c_P)
2. under **constant volume** (c_V).

For a **perfect, monoatomic gas**, $u = \frac{\frac{3}{2}nkT}{\varrho} = \frac{3}{2}\frac{\Re}{\mu}T$, and

- $c_V = (\frac{dq}{dT})_V = (\frac{du}{dT})_V = \frac{3}{2}\frac{\Re}{\mu}$
- Can show $c_P - c_V = \frac{\Re}{\mu}$ (for derivation see e.g. Kippenhahn section 4.1)

Then $c_P = \frac{5}{2}\frac{\Re}{\mu}$ and $\boxed{\boldsymbol{\gamma_{ad} \equiv c_P/c_V}}$ **= 5/3** γ_{ad}=**"adiabatic exponent"** or **"heat capacity ratio"**

For a photon gas or a relativistic particle gas, can show γ_{ad}**= 4/3.** (Exercise to derive)

Virial theorem

Hydrostatic equilibrium:

$$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4}$$

Muliply by $4\pi r^3$and integrate

$$\int_0^R 4\pi r^3 \frac{dP}{dm} dm = -\int_0^R 4\pi r^3 \frac{Gm(r)}{4\pi r^4}\, dm$$

Integrate LHS by parts

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$[4\pi r^3 P]_0^R - \int_0^R 12\pi r^2 \frac{dr}{dm} P\, dm = 0 - 0 - \int_0^R \frac{3P}{\rho} dm$$

For a perfect gas

$$\frac{P}{\rho} = \frac{\mathfrak{R}}{\mu} T = (c_P - c_V)T = (\gamma_{ad} - 1)c_V T$$

For monoatomic case γ_{ad} =5/3 and, with u the internal energy per unit mass:

$$\frac{P}{\rho} = \frac{2}{3} u \qquad \text{and then LHS} = -2E_{int}$$

where E_{int} is the total internal energy.

RHS:

$$-\int_0^R \frac{Gm}{r} dm = E_{grav}$$

So

$$\boxed{E_{grav} = -2E_{int}}$$

Note to derive this form we have assumed

1) Hydrostatic equilibrium
2) Perfect monoatomic gas

The virial theorem is useful e.g. to determine
the thermal evolution time-scale of a star (next slide).

Time-scales relevant for stellar evolution

Nuclear time-scale:

$\tau_{nuc} = \frac{E_{nuclear-reservoir}}{\varepsilon M_{nuclear-reservoir}}$ 10^{11} years for H-burning in the sun. A few hours for Si burning in a massive star.

Energy release rate per unit mass

Hydrodynamic time-scale:

$\tau_{hydro} = \sqrt{\frac{R}{g}} = \left[g \sim \frac{GM}{R^2}\right] = \sqrt{\frac{R^3}{GM}} \sim \frac{1}{2\sqrt{G\bar{\rho}}}$ 30 minutes for the sun ($\rho \sim 1$ g cm^{-3}). 1 second for iron core in massive star ($\rho \sim 10^8$ g cm^{-3}).

Kevin-Helmholtz time-scale (time-scale for significant structural (radius) change in a star when gravitational contraction provides the luminosity):

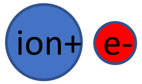
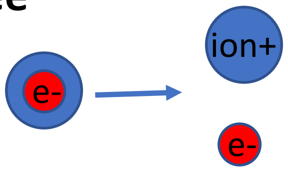
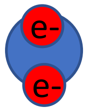
$$\tau_{KH} = \frac{E_{int}}{L} = \frac{-E_{grav}/2}{L} \approx \frac{GM^2}{2RL} = 1.5 * 10^7 \mathrm{y} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1}$$

Virial theorem

Thermal adjustment time-scale (time for a thermal fluctuation to spread through the star). Can show

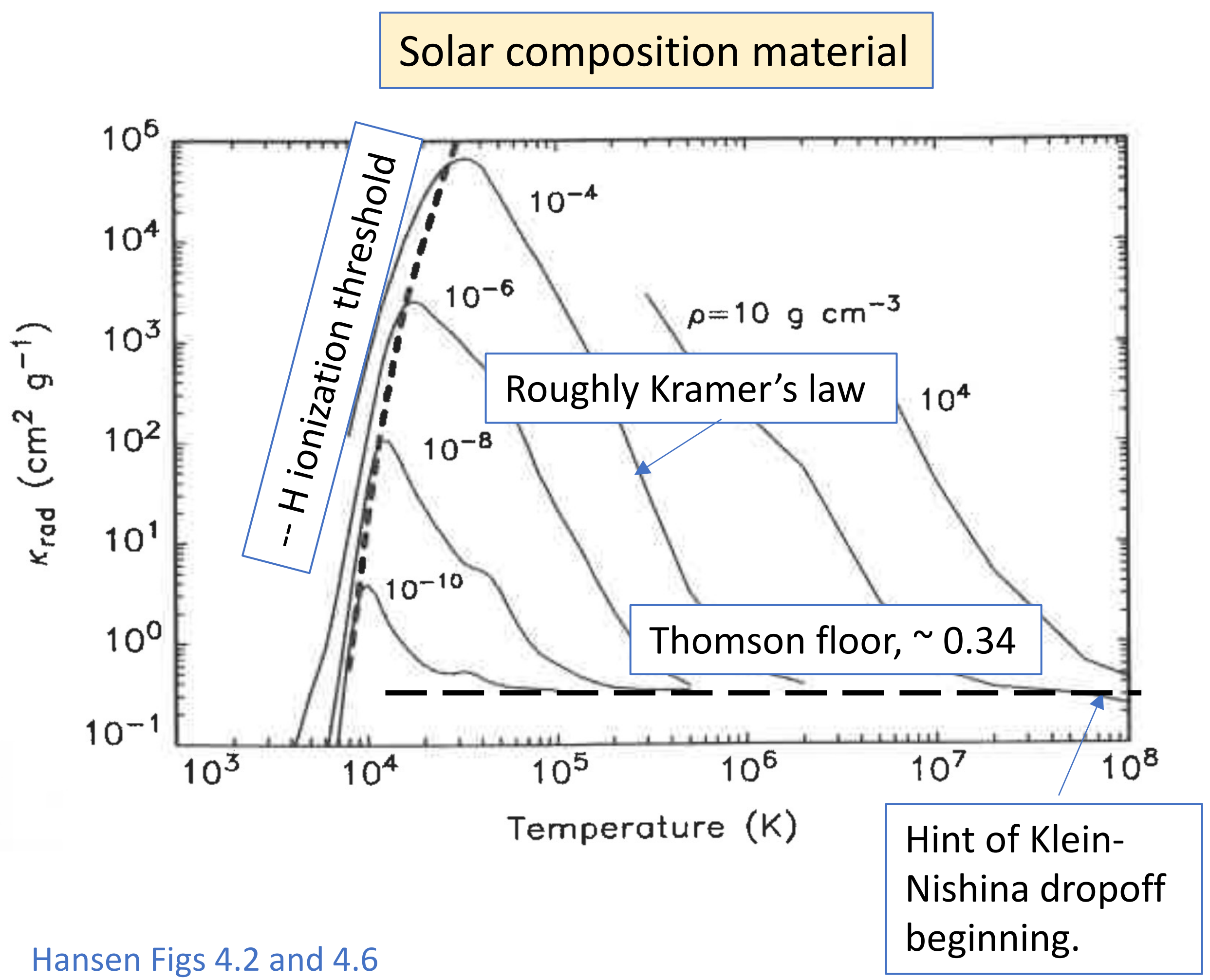
$\tau_{thermal} \sim \tau_{KH}$

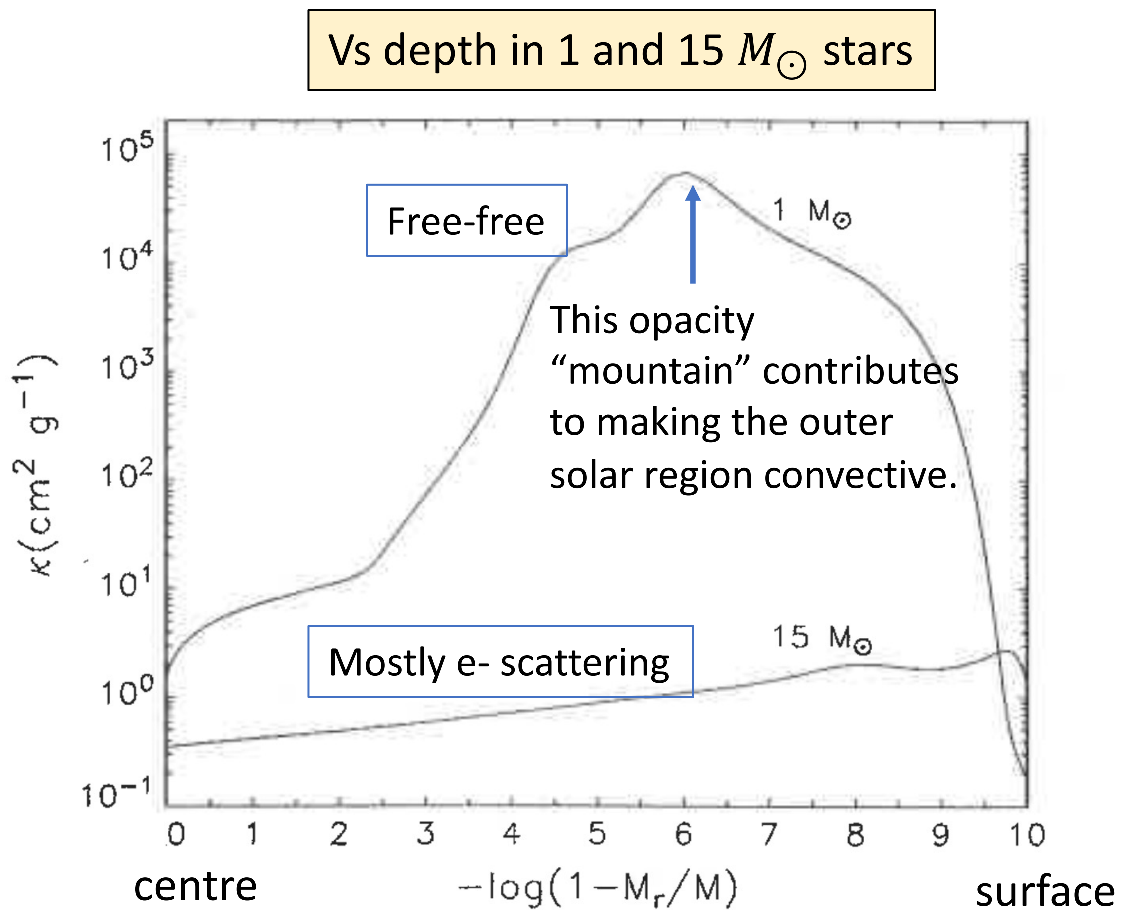
Opacity

Type	Rosseland κ [cm^2/g]	Comments
Electron scattering 	$0.4\mu_e^{-1}$	Klein-Nishina corrections at $T \gtrsim 10^8$K (lowers κ). Note no (ρ, T) dependency otherwise if gas is fully ionized.
Free-free  *An electron-ion pair close enough Together can absorb a photon.*	$\approx 1.0\rho\left(\frac{T}{4E6\ K}\right)^{-7/2}$ $\bar{Z}^2\mu_e(\rho,T)^{-1}\ \mu_{ion}(\rho,T)^{-1}$	All opacities of form $T^{-7/2}$ called “Kramer opacity”.
Bound-free 		Typically a ν^{-3}dependency of cross-section which is the same as free-free → gives same $T^{-7/2}$ form when Rosseland-integrated for single ion and $kT \gg$ ionization threshold. Special topic: H$^-$ ion (important e.g. in sun’s surface layers)
Bound-bound 		In addition to atomic lines, molecular lines can be important at low T ($\lesssim 5000\ K$ not because molecules are very abundant but because have many states. Very hard to model molecules and dust → opacities at low temperatures very uncertain and this is an obstacle to accurately model cool, evolved envelopes. Modern tables include effect of billions of transitions.

In stellar evolution models **tables** are used where opacity is looked up as function of $T, \rho,$ and composition.

Opacity





Hansen Figs 4.2 and 4.6

Understanding the MS – the starting point for late stellar evolution

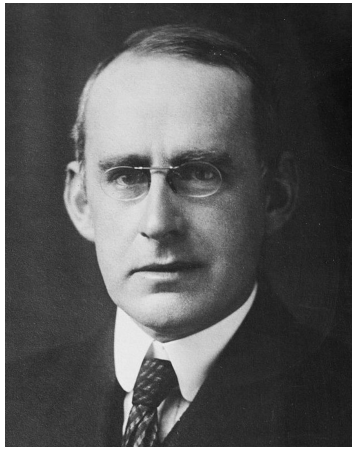
With a diffusion time expression, we can relate *L* and *M* for a star

Assume ideal gas: $P = \frac{\Re}{\mu}\rho T$

universal gas constant (8.315E7 erg/K/g)

c subscript denotes values in core

Hydrostatic equilibrium: $\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \rightarrow P_c \sim \frac{GM\rho_c}{R}$



Then

$$T_c \sim \frac{GM\mu}{R\Re} = 1.2 * 10^7\ \mathrm{K}\left(\frac{\mu}{0.5}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}$$

Now approximate luminosity ~ internal radiative energy / diffusion time →

Take characteristic T as mean of T_c and T_{surf} ~ $T_c/2$.

$$L = \frac{\frac{4\pi R^3}{3} a \left(\frac{T_c}{2}\right)^4}{\frac{27}{4\pi c}\frac{\kappa M}{R}} = 1 * 10^{34}\left(\frac{\kappa}{0.4\ \mathrm{cm^2 g^{-1}}}\right)^{-1}\left(\frac{\mu}{0.5}\right)^4\left(\frac{M}{M_\odot}\right)^3 \text{ erg s}^{-1}$$

Compare $L_\odot = 4 * 10^{33}$ erg s^{-1}

Note: **The luminosity depends only on mass and opacity (and μ), not on the energy generation rate!** Discovered by Arthur Eddington before the energy generation by nuclear reactions was understood. $L \propto M^3$ derived without knowing how stars work!

How does *R* depend on *M* (on MS)?

Turns out one can only get to this by quite complex derivations using so called “homology relations” (e.g. Kippenhahn chapter 20). With an energy generation function $\varepsilon \sim T^{\nu}$, a perfect gas law, and constant opacity one may derive

$$\boldsymbol{R \sim M^{\frac{\upsilon-1}{\upsilon+3}}}$$

Pp chain: $\upsilon\sim 4 \rightarrow R\sim M^{0.4}$

CNO cycle: $\upsilon\sim 16 \rightarrow R\sim M^{0.8}$

Then R^2 grows only with M as a 0.8 - 1.6 power law.

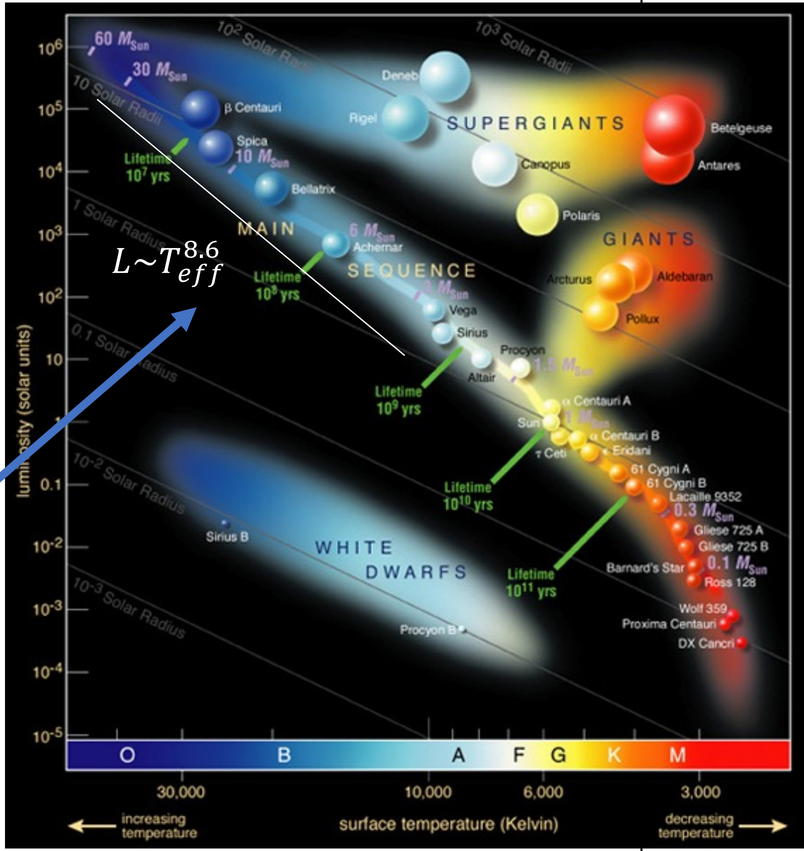
From the previous slide we have $M\sim L^{1/3}$. Then $R\sim L^{\frac{(\upsilon-1)}{3(\upsilon+3)}}$. We can then write

$$T_{eff} = \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4} \sim L^{\frac{1}{4}-\frac{\upsilon-1}{6(\upsilon+3)}}$$

$\upsilon = 4$: $L\sim T_{eff}^{5.5}$

$\upsilon = 16$: $L\sim T_{eff}^{8.6}$

Good agreement with observed main sequence.

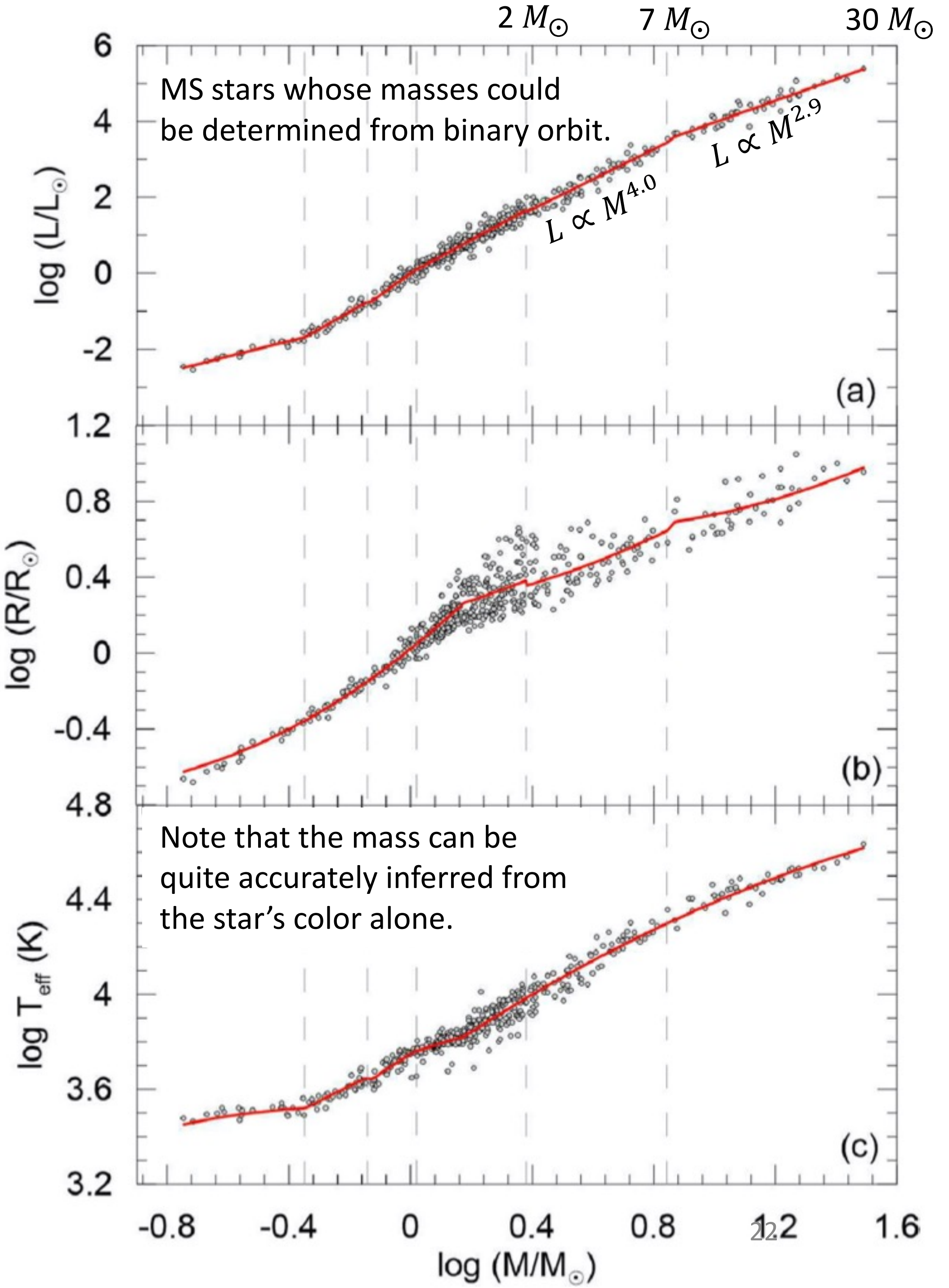


For post-main sequence stars the homology ansatz is poor and these relations less accurate.

Through eclipsing binaries one can determine *M* from orbital properties and check the basic idea: <u>mass is the dominant factor for a star's properties on the MS</u>

Same *M* can still give somewhat different L, T_{eff} due to

- Different age (e.g. sun's *L* evolves factor 2 over its MS time).
- Different metallicity (the plotted sample covers ~ 0.1-2 times solar).
- Different rotation.



Eker+2018

The Eddington luminosity : sets the maximum mass of stars

Acceleration of a shell:

$$a = a_{grav} - a^{gas}_{pressure-gradient} - a^{radiation}_{pressure-gradient}$$

Any hydrostatic equilibrium ($a = 0$) needs $a^{radiation}_{pressure-gradient} < a_{grav}$, or

$$\frac{1}{\rho}\frac{dP_{rad}}{dr} < \frac{Gm}{r^2}$$

$\frac{dT}{dr} = \frac{3\kappa\rho l}{4acT^3}$ (will be derived later, l is luminosity at mass coordinate m)

$$P_{rad} = \frac{1}{3}aT^4 \rightarrow LHS = \frac{4}{3\rho}aT^3\frac{dT}{dr} = \frac{\kappa l}{4\pi r^2 c}$$

Then, for $m = M$ ($l = L$):

$$\boxed{L < \frac{4\pi cGM}{\kappa}} \longrightarrow L < 3.8 * 10^4 L_\odot * \left(\frac{M}{M_\odot}\right)$$

for $\kappa = 0.34$ cm^2 g^{-1}

Use now $L \approx L_\odot\left(\frac{M}{M_\odot}\right)^3 \rightarrow M_{max} \sim 195\ M_\odot$

Stellar evolution processes and equations

Stellar structure equations (hydrostatic limit)

Four differential equations plus a scheme for convection: | Physical law: | Example/comment:

Note MESA has also time-dependent terms in Eqs (2) and (3).

1 $$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

Mass conservation

2 $$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4}$$

Momentum conservation (for $\tau_{nuc}, \tau_{KH} \gg \tau_{hydro}$)

Special case (hydrostatic limit) of an equation of motion.

3 $$\frac{d(l_{rad} + l_{conv})}{dm} = \varepsilon^*(T, \rho, x_1, x_2, ..) - \eta_\nu(T, \rho, x_1, x_2, ..)$$

Composition

Nuclear burning power

Neutrino cooling

Energy conservation (for $\tau_{nuc} \gg \tau_{KH}$)

*Exclude the fraction of nuclear energy release emitted as neutrinos.

4 $$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{l_{rad}}{(4\pi r^2)^2}$$

5 l_{conv} = Mixing-Length-Theory sol.

Energy transport

l_{rad} is the radiative transport rate (diffusion limit), Can include also conduction (rarely needed).

l_{conv} is the convective transport rate.

- **Lagrangian form** (mass as independent variable rather than radius).
- Want to solve for the 6 quantities {r, ρ, P, T, l_{rad}, l_{conv}} as function of mass coordinate for given current composition $x_1, x_2, ...$An **EOS** comes in as the sixth physical law to close the system.
- **Additional constitutive relations** needed for: opacity κ, nuclear burning rate ε, heat capacity c_P (enters MLT). All these (and the EOS) depend only on **local quantities $(\rho, T, x_1, x_2, ..)$, no derivatives**.

Composition equations

$$\frac{dx_i}{dt} = \left(\frac{dx_i}{dt}\right)_{\text{nuclear}} + \left(\frac{dx_i}{dt}\right)_{\text{diffusion}}$$

Includes convective mixing, which is treated as as diffusion process (more later). "Normal" diffusion typically unimportant.

In the hydrostatic approximation, may solve composition equations separately from the structure equations (iterate Initial Value Problem solution for the composition with Boundary Value Problem for the structure). However, MESA solves the equation blocks fully coupled which is more general and natural – good efficiency is retained by innovative time step selection algorithms. (Paxton+2011)

Detailed nucleosynthesis results (e.g. s-process) are often done as a post-processing step : for the evolution of the star's structure one needs only species relevant for the energy generation.

Simple solutions to the stellar structure equations : polytropes

Make ansatz that pressure is uniquely defined by the density only, and of form

$$P = K\rho^{\gamma} \equiv K\varrho^{1+\frac{1}{n}}.$$

Polytropic constant K

Polytropic exponent γ

Polytropic index $n = 1 / (\gamma - 1)$

Care the polytripic exponent γ
is not the same as the adiabatic exponent γ_{ad} :
But they both often take on values 5/3 and 4/3.

Then, don't need to solve for temperature, which in turn allows to skip the (sometimes complicated) equations for energy conservation and transport : eqs 3, 4 and 5 on slide 18.

Turns out such an ansatz is useful for quite a broad range of applications, e.g. (derivations later),

- Degenerate, non-relativistic gas: $P \propto \rho^{5/3}$ $(n = 3/2)$
- Degenerate, relativistic gas: $P \propto \rho^{4/3}$ $(n = 3)$

K is a known constant

- Fully convective, fully ionized, ideal gas: $P \propto \rho^{5/3}$ $(n = 3/2)$
- Isothermal, fully ionized gas: $P \propto \rho$ $(n = \infty)$

K is a free parameter

Polytropes are not just theoretical curiosities, they are often used in many types of simulations of pre-supernova stars when large parameter spaces need investigating.

Energy transport

Internal energy can be transported by three modes: **radiation**, **conduction** and **convection.** Let $l(m)$ represent this <u>total</u> energy flux (erg/s).

In hydrostatic equilibrium, $PdV = 0$ and any change in total energy flux over two surfaces bounding a mass shell must be due to internal energy being created inside the shell by nuclear reactions, minus any neutrino cooling (as neutrinos escape freely and we don't bookkeep them):

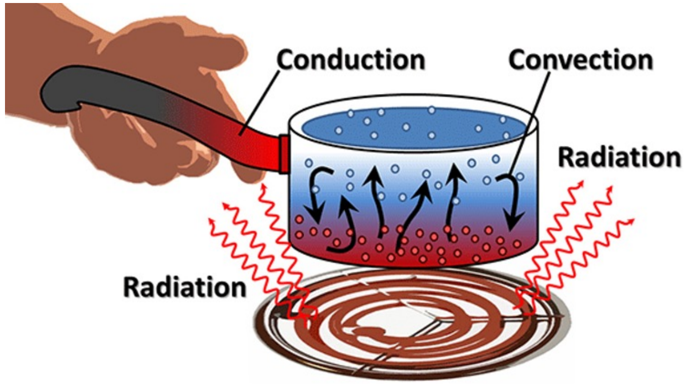
$$\frac{dl}{dm} = \varepsilon^* - \eta_\nu$$

Where ε^* is the nuclear energy generation rate (excluding neutrinos) and η_ν is the neutrino cooling rate.

Conduction can, by certain choice of variables, be expressed as a type of radiative transport. Therefore one often breaks down

$$l = l_{rad} + l_{conv}$$

l_{rad} may include conductive transport.



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Radiative energy transport

The mean-free path is

$$\lambda_{mfp} = \frac{1}{\kappa\rho} \sim \frac{4\pi}{3}\frac{R^3}{\kappa M} = 2\ \mathrm{cm}\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{R}{R_\odot}\right)^{3}$$, for $\kappa = 0.34$ cm^2 g^{-1}

The temperature change over a mean-free-path is, roughly

$$\Delta T \sim \frac{T_c}{R}\lambda_{mfp} = 10^{-4}\ \mathrm{K}\left(\frac{T_c}{10^7\ K}\right)^{-1}\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{R}{R_\odot}\right)^{2}$$

This is completely neglegible and the radiation field will therefore be **well described by a blackbody at a well-defined local temperature**. Then, the **radiation energy density** is

$$U = aT^4$$ (erg cm^{-3}) ,where $a = 7.56 * 10^{-15}$ erg cm^{-3} K^{-4}

The short λ_{mfp} also means transport of photons will be well described as a diffusion process, which has the general form

$$j = -D\nabla n$$

where *j* is the flux of particles (cm^{-2} s^{-1}), *n* is the particle number density (cm^{-3}) and

$D = \frac{1}{3}v\lambda_{mfp}$ is the **coefficient of diffusion** (*v* is the particle speed), unit cm^2 s^{-1}.

Radiative energy transport

If we count flux of energy instead of flux of particles (just multiply both sides by the mean photon energy), we get:

$$F_{rad} = \frac{1}{3}\frac{c}{\kappa\rho}\frac{dU}{dr} = \frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

Inserting $F_{rad} = \frac{l_{rad}(m)}{4\pi r^2}$,

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho l_{rad}}{r^2T^3}$$

or, with $dm = 4\pi r^2\rho dr$, equivalently

$$\frac{dT}{dm} = \frac{3}{64\pi^2 ac}\frac{\kappa l_{rad}}{r^4T^3}$$

which is the form used in the stellar structure equation set.

Conductive opacity

Conduction involves energy transport by electrons moving through the material, hitting other electrons and ions along the way. They diffuse down a density/temperature gradient just as photons → basically the same process just for two different particles. Conduction can be important in white dwarfs.

- Energy transport by conduction can become important at very high density e.g. in white dwarfs.

- In general, $F_{cond} = k_{cond} \nabla T$ where k_{cond} is the **coefficient of conduction** (unit erg K^{-1} cm^{-1} s^{-1})

$$= \frac{4acT^3}{3\rho} \frac{k_{cond}}{\frac{4acT^3}{3\rho}} \nabla T \equiv \frac{4acT^3}{3\rho} \frac{1}{\kappa_{cond}} \nabla T$$

Then

$$F_{rad} + F_{cond} = \frac{4acT^3}{3\rho} \left(\frac{1}{\kappa_{cond}} + \frac{1}{\kappa_{rad}} \right) \nabla T$$

F_{cond} represents energy flow by particles, F_{rad} by photons.

- Clearly, if κ_{cond} becomes smaller than κ_{rad}, energy is transported by conduction more efficiently than by radiation.

Energy transport by convection.
Prelude: Nabla (or del) notation

∇ ("nabla" or "del") denotes, in stellar structure tradition, the dimensionless derivative of temperature with respect to pressure as one moves in depth:

$$\nabla \equiv \frac{\frac{dT}{dm} / \frac{dP}{dm}}{T/P}$$

Because $d \ln X = \frac{dX}{X}$, one can also write this as

$$\nabla \equiv \frac{d \ln T}{d \ln P}$$

On this form one has to remember that nablas do not refer to (local) EOS derivatives but derivatives along a depth movement.

Similarly, the quantity

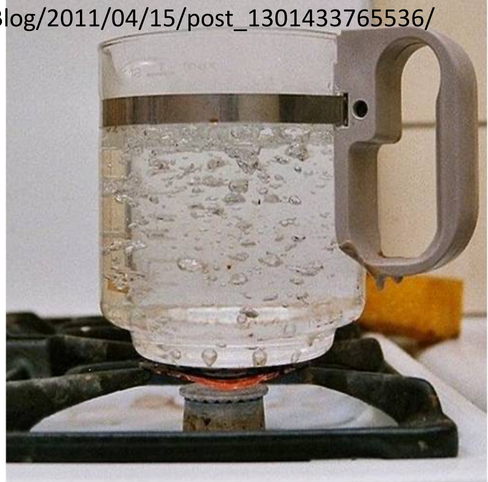
$$\nabla^{\mu} = \frac{d \ln \mu}{d \ln P}$$

refers to a dimensionless μ gradient.

All textbooks and most articles use these classic notations so good to know them.
$\nabla = \nabla_T$ in the MESA papers.

Energy transport by convection

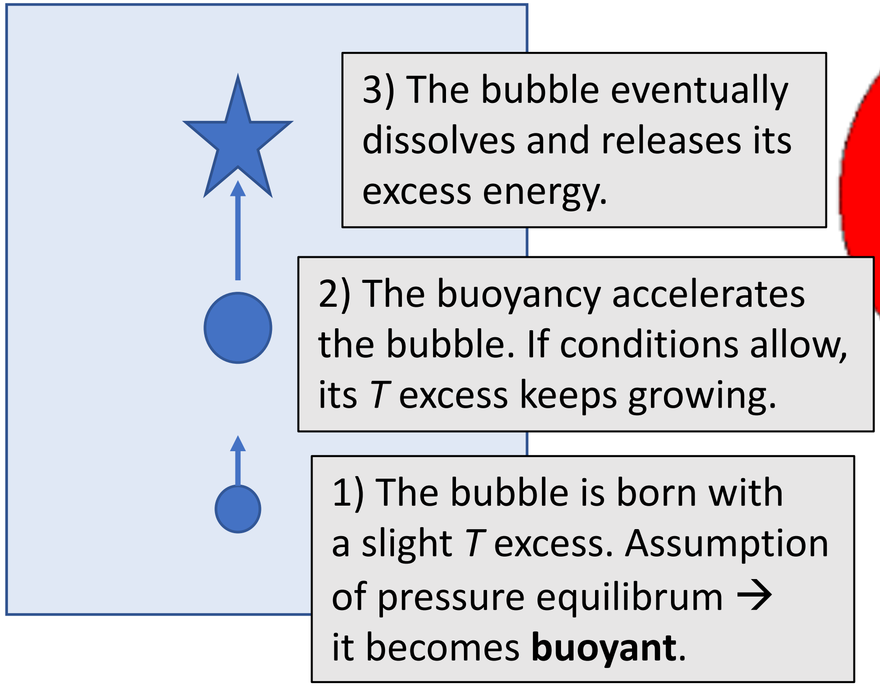
https://blogs.nasa.gov/ISS_Science_Blog/2011/04/15/post_1301433765536/

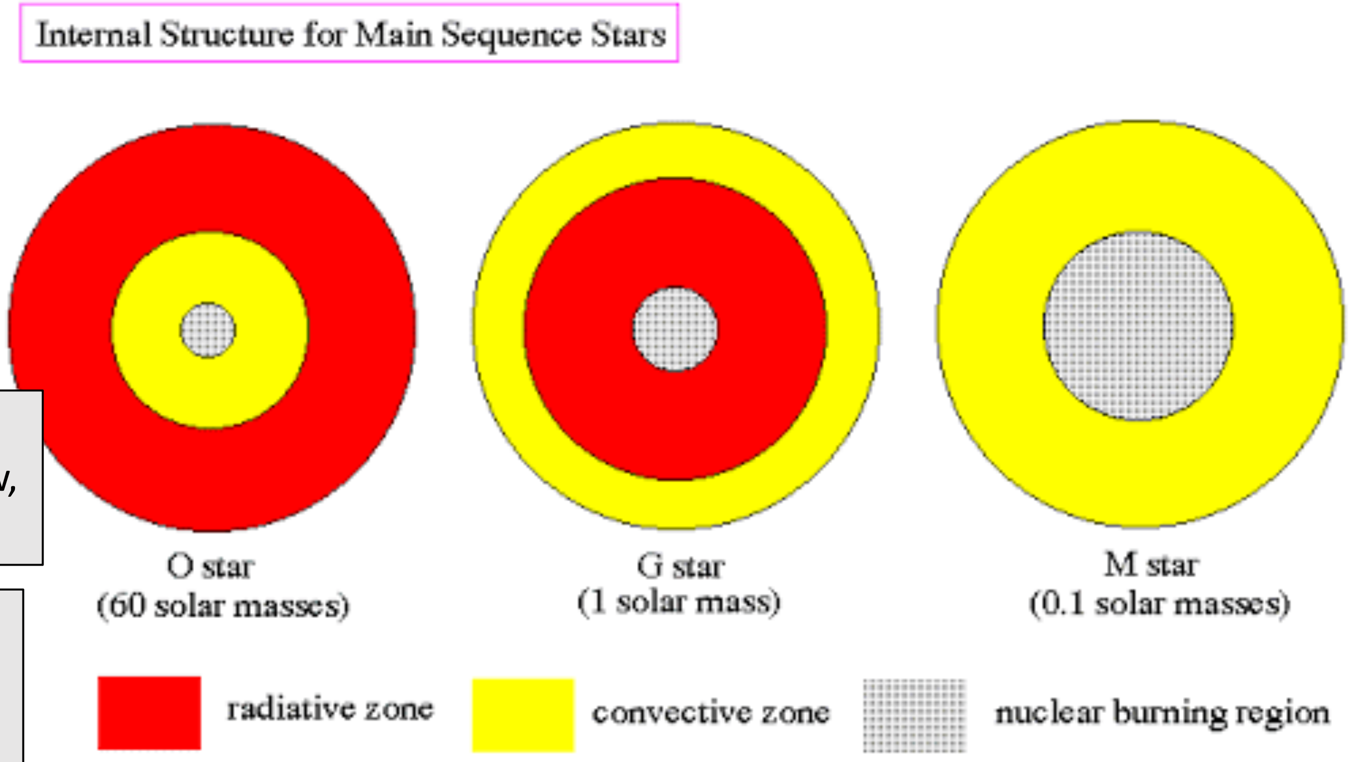


In stars energy transport by convection is a key process.

It arises from a multi-dimensional instability that allows hot bubbles to form, accelerate, and transport internal energy upwards towards the star's surface.

In theoretical frameworks, the origin of the fluctuations/bubbles is not specified, neither is the dissolution mechanism.





http://abyss.uoregon.edu/~js/ast122/lectures/lec14.html

Energy transport by convection: Conditions for on-set

Dynamic stability: is a temperature-perturbed blob returned towards its origin when starting to move by buoyancy?

Note these gradients are negative

Start from requiring $(d\rho/dr)_{blob} > (d\rho/dr)_{surr}$. "The blob's density decreases slower than the density of its surroundings as it moves upwards, so it will soon become too dense to stay buoyant".

Ansatz: Pressure equilibrium with surroundings : corresponds to subsonic bubble motions.

An EOS can generally be written as

$$d\rho = \frac{\delta\rho}{\delta P}dP + \frac{\delta\rho}{\delta T}dT + \frac{\delta\rho}{\delta\mu}d\mu \rightarrow \frac{d\rho}{\rho} = \underbrace{\boxed{\frac{P}{\rho}\frac{\delta\rho}{\delta p}}}_{\equiv\alpha}\frac{dP}{P} + \underbrace{\boxed{\frac{T}{\rho}\frac{\delta\rho}{\delta T}}}_{\equiv -\delta}\frac{dT}{T} + \underbrace{\boxed{\frac{\mu}{\rho}\frac{\delta\rho}{\delta\mu}}}_{\equiv\varphi}\frac{d\mu}{\mu}$$

From this the stability criterium can be transformed to (Exercise Set 1):

$$\nabla < \nabla_{blob} + \frac{\varphi}{\delta}\nabla^{\mu}$$

What happens in practice is that one often first tries a solution without convection, which then gives a solution for the gradient when only radiative transport is considered, $\nabla = \nabla_{rad}$. If instability is then indicated, one "turns on" convection (see later). Further, one can show $\nabla_{blob} \geq \nabla_{ad}$ (and in some theory variants these are assumed equal) so the stability test can be written

Gradient of blob under an adiabatic motion.

$$\boxed{\nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta}\nabla^{\mu}}$$

Ledoux stability criterion

$$\boxed{\nabla_{rad} < \nabla_{ad}}$$

Schwarzschild stability criterion
(specific case of homogenous composition)

"How does the (adiabatic) blob's temperature (∇_{ad}) vary with position compared to the surroundings (∇_{rad})"? Its motion is damped out only if it varies faster (cools faster) and thus loses its buoyancy.

Note that ∇^{μ} can be positive or negative, but is normally positive (heavier elements typically lie deeper in the star where pressure is higher). The Ledoux stability criterium is then more easily satisfied than the Schwarzschild one
→ ***molecular weight gradients stabilize.***

If both stability conditions are satisfied there is <u>**no**</u> convection and energy flux is carried by radiation and conduction only. If neither is satisfied **convection is active** and carries some part of the energy flux (to be determined). If Ledoux is satisfied but not Schwarzschild, we have a situation called **semi-convection**.

How to get ∇_{rad} and ∇_{ad}

Given a total luminosity l_{tot} (including both radiative and convective transport), and assuming hydrostatic equilibrium, the gradient that would mean that all energy is transported by radiation is

$$\boxed{\nabla_{rad} \equiv \frac{3}{16\pi acG}\frac{\kappa l_{tot}(m)P}{mT^4}}$$

Note for radiation pressure limit, $\nabla_{rad} = C * \kappa * \frac{l(m)}{m} = 6 * 10^{-5}\left(\frac{l(m)}{L_\odot}\right)\left(\frac{m}{M_\odot}\right)^{-1}$

What about ∇_{ad}? The derivation is lengthy (see e.g. Kippenhahn Sec 4.1) but rests on just the first law of thermodynamics. One gets:

$$\boxed{\nabla_{ad} \equiv \left(\frac{P}{T}\frac{dT}{dP}\right)_s = \frac{P}{T}\frac{\delta}{\rho c_P}}$$

s subindex means at constant entropy = adiabatic motion

So, with formulas for ∇_{rad} and ∇_{ad} , we can now determine whether a certain structure will start to convect or not. In the Schwarzschild case the answer depends only on local state variables ρ, P, T, c_P, and the EOS (δ part), in the Ledoux case also on the composition gradient ∇^{μ} .

The quantitive framework used to calculate the convective energy flux is **mixing-length-theory** (next section).

Energy transport by convection : Mixing-Length-Theory (MLT)

Make ansatz that convective energy flow is carried by a set of identical blobs that move a certain distance l_m (the ***mixing length***) before dissolving. The geometry (typically spheres) and size (typically set to equal l_m) of these are ingoing assumptions. **If we can calculate the velocity *v* and temperature excess ΔT in such blobs, we can determine their energy transport**.

The value of l_m is determined from calibration to observations. Normally one expresses it in units of **pressure scale heights** $H_P = -dr/d\ln P$(unit cm):

$\alpha_{MLT} = l_m/H_P$ (dim-less) Typically $\alpha_{MLT} = 1.5 - 2$ from stellar calibrations. **Single free parameter of MLT framework.**

Derivation. Put up equations for momentum and internal energy evolution of a blob. Assume initially small perturbations in temperature and velocity, $\frac{\Delta T_0}{\Delta T} \ll 1$ and $\frac{v_0}{v} \ll 1$, so ΔT and v arise by buoyancy evolution and their values at bubble burst time is independent of initial fluctuation properties.

One blob of mass M releases $M * C_P * \Delta T$ of energy (erg) when it bursts. If the number density of blobs is n and they move with velocity v, then

$F_{conv} = f\rho v c_P \Delta T$, where f is a volume filling factor assumed to be ~1.

In the general case blobs are allowed to exchange energy with the surroundings by radiative diffusion also.

Energy transport by convection : Mixing-Length-Theory (MLT)

In practice:

Find numerically the root to an equation relating ∇ to local quantities $(P, T, l, \rho, \kappa, l_m, EOS, C_P)$ and one spatial derivative (H_P).

Limits:

Strongest possible convection: $\nabla \rightarrow \nabla_{ad}$

Weak convection: $\nabla \rightarrow \nabla_{rad}$

In a region with some degree of convection, ∇_{rad} is always larger than ∇_{ad} , so partially effective convection always gives a gradient in between these limits:

$\nabla_{ad} < \nabla < \nabla_{rad}$.

Energy transport by convection : Mixing-Length-Theory (MLT)

Examples:

Convective (outer) layers in the sun:

$\nabla = \nabla_{ad} + 10^{-12}$

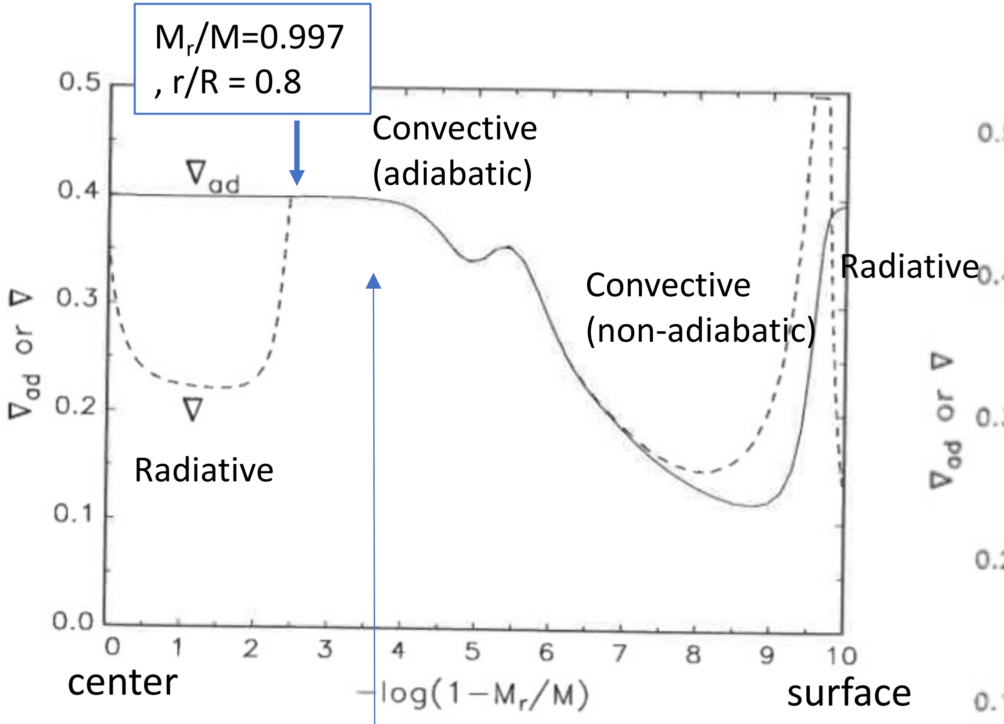
$\Delta T \sim 0.1$ K

$v \sim 10$ km s^{-1}
(=10^{-5} times the sound speed)

Lifetime of a blob ~10d.

It turns out often $\nabla \rightarrow \nabla_{ad}$, and then the details of the theory don't matter : there is a correctly captured limiting behaviour.

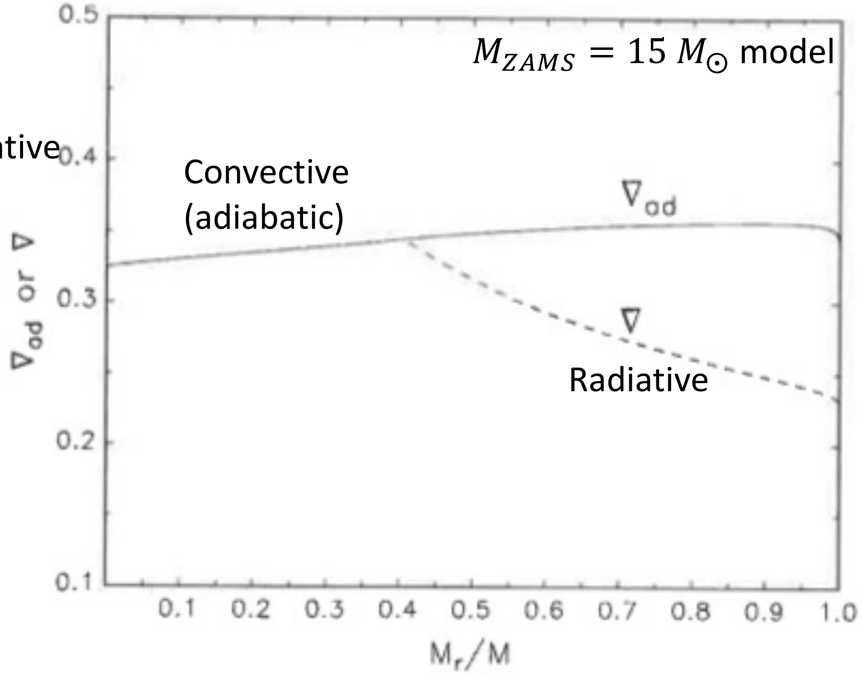
Low-mass stars



The sun gets a very high κ in outer layers which makes radiative diffusion inefficient and convection is triggered.

High-mass stars

$$\nabla_{rad} \equiv \frac{3}{16\pi acG} \frac{\kappa l_{tot}(m)P}{mT^4}$$



Massive stars have steep T-dependency for energy generation (e.g. T^{16} for CNO-cycle). This leads to very compact energy generation centra (high l_{tot}/m) and radiative diffusion is unable to transport all this flux.

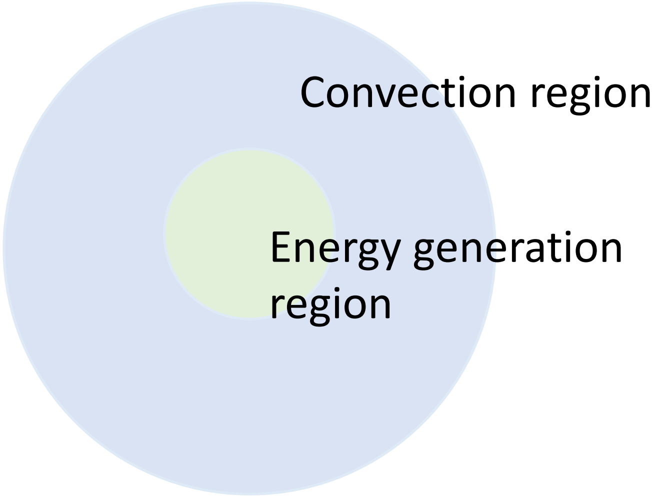
Hansen Figs 5.2 and 5.6

Convection: compositional mixing

Convection normally mixes matter so quickly and efficiently that a convective zone has a homogenous composition at any given time: e.g. <u>changes in nuclear composition arising in the inner (small) burning core rapidly spread throughout the whole (large) convective core.</u>

For detailed modelling, compositional mixing is treated with a diffusion equation, with velocity from the MLT solution:

$$D_c^{conv} = \frac{1}{3} v \overbrace{\alpha_{MLT} H_P}^{\text{Mean-free path of convective blobs}}$$, unit cm^2 s^{-1}



Use this diffusion coefficient in

$$\left(\frac{dx_i}{dt}\right)_{diff} = \frac{\delta}{\delta m}\left[(4\pi r^2 \rho)^2 D_c^{conv} \frac{\delta x_i}{\delta m}\right]$$

Note that D_c^{conv} is usually many orders of magnitudes larger than D for particle diffusion. In general, compositional mixing by real diffusion is inefficient and does not need to be considered except in some very particular/special cases.

Convection: Overshooting

The Ledoux and Schwarzschild criteria are local. But the motion of a convective blob is a trajectory over a finite distance –> what happens, exactly, at the border between stable and unstable layers? Here blobs could arrive with a significant velocity and "overshoot" before decelerating and breaking up → further transport of both heat and composition over the border into the formally stable region.

Two common approaches:

1. **Extend the convective region with some length** $\Delta r_{ov} = \alpha_{ov} H_P$. Often $\alpha_{ov} \backsim 0.2$ is used, can be calibrated to some extent by model-data comparisons.
2. **Add a compositional diffusive mixing term that tapers off into the stable region,** $D_c^{ov}(r) = D_c^{conv,border} * exp\left(-2|r - r_{border}|\right)/f_{ov}H_p\big)$. From Herwig 1997 who derive $f_{ov} \approx 0.02$ from multi-D simulations of Freytag+1996 combined with reproducing the observed MS width. However, in other contexts quite different values are needed (see later). MESA uses this method, and allows the use of different f_{ov} at different boundaries.

$D_c^{conv,border}$ is the convective diffusion coefficient just inside the stability boundary (obtained from MLT solution).

Importance of overshooting:

- MS lifetimes and luminosities.
- CO core masses.
- Late shell mergers in massive stars.

Convection: semi-convection

When Ledoux stability is fulfilled but Schwarzschild is not (the molecular weight gradient is needed to prevent convection) → can show that in this situation oscillations grow with time → matter mixes somehow by "vibrational instability", "**semi-convection**"

Treatment: Diffusion coefficient for compositional mixing with:

1. $D_c^{semi} = C * D_{rad} D_c^{conv} / (D_{rad} + D_c^{conv})$ (KEPLER, $C = 0.1$) Weaver+1978 $D_{rad} = 1/3 \frac{acT^3}{\kappa\rho^2}\left(\frac{du}{dT}\right)^{-1}$
2. $D_c^{semi} = \alpha_{sc}(K_r/6C_P\rho) * (\nabla - \nabla_{ad}) / \left(\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\nabla^{\mu}\right)$ (Langer 1983,1985, used in MESA). K_r=radiative conductivity=$4acT^4/3\kappa\rho$. $\alpha_{sc} \sim 0.1$.
3. $D_c^{semi} = (D_{rad} * D_{ionic})^{1/2} * (\nabla_{rad} - \nabla_{ad}) / \frac{\varphi}{\delta}\nabla^{\mu}$ Spruit 1992 For D_{ionic} see Eq. 61 in this paper

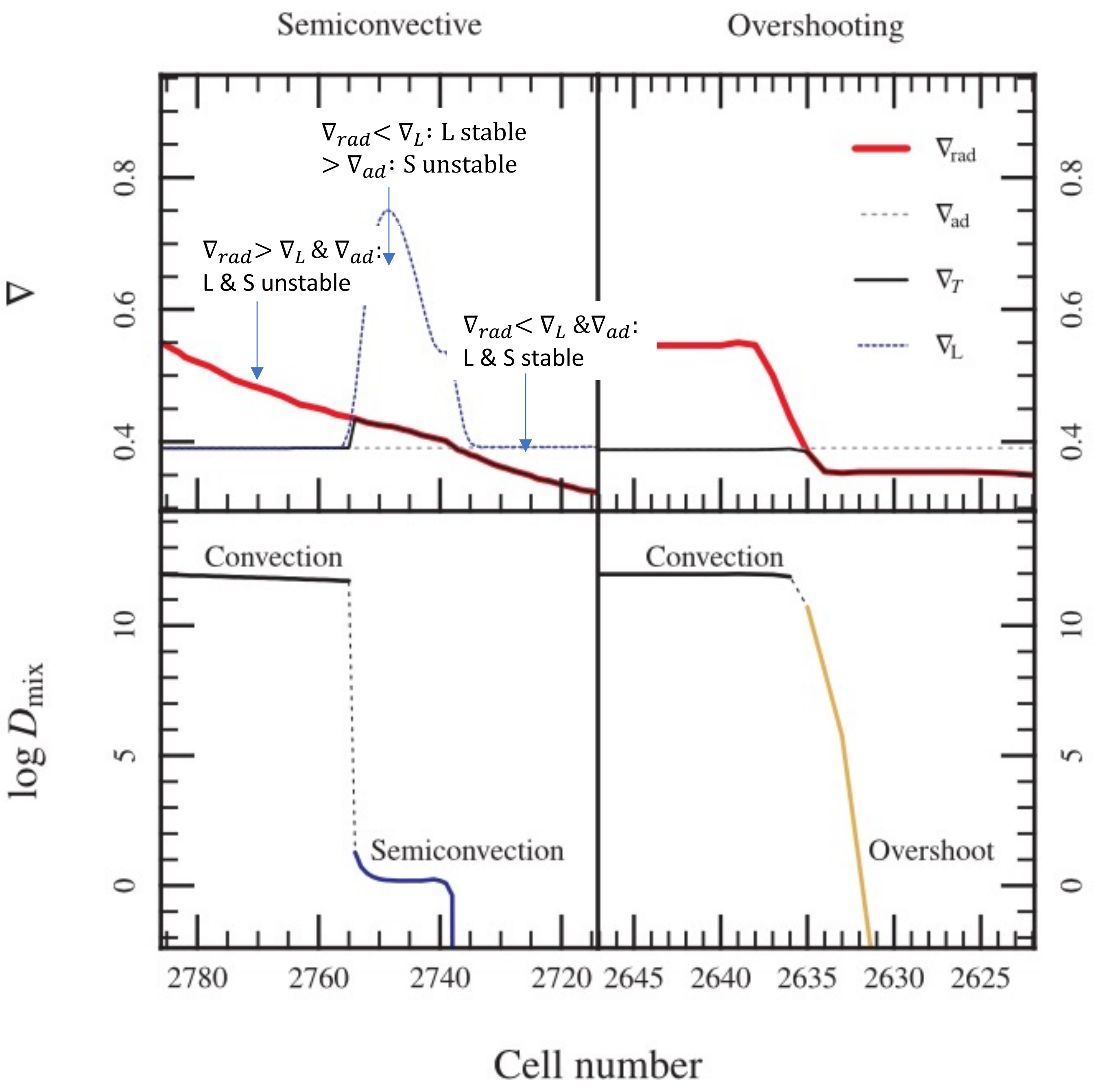
Different recipes can give D_c^{semi} varying by several orders of magnitude : semiconvection is an essentially unsolved problem in astrophysics.

Semi-convection is typically unimportant for energy transport but can be important for composition mixing. Semi-convection is related to compositional gradients and massive stars develop significant such in their cores at late times.

Important mainly for post-MS evolution:

- Shell H burning and phase when the RSG structure is developed (early or late in He burning). Inefficient semi-convection favours "blue loops" in the HR diagram.
- Core He burning and mass of final CO cores.
- Core silicon burning: electron captures lead to significant μ gradients and can give semi-convective layers.

Example of overshooting and semi-convection



Notion here:

$$\nabla_T = \nabla$$

$$\nabla_L = \nabla_{ad} + \frac{\varphi}{\delta}\nabla^{\mu}$$

Paxton 2013 (MESA paper 2) Fig 11

Equation of state

Radiation pressure: $P_{rad} = \frac{1}{3}aT^4 = 2.5 * 10^{10}\left(\frac{T}{10^7 K}\right)^4$ dyne cm^{-2}

Internal energy:

$U = 3P$

Gas pressure, ideal gas approximation: $P_{gas} = nkT = \frac{\Re}{\mu}\rho T = 8.3 * 10^{14}\mu^{-1}\rho\left(\frac{T}{10^7\ K}\right)$ dyne cm^{-2}

$U = \frac{3}{2}P + E_{ion.-potential}$

Need to know ionization state (Saha, plus possibly consideration of pressure ionization)
Holds also for relativistic particles.

Electron degeneracy pressure:

$T = 0$ limit: $P_{deg} = 6 * 10^{22} * (x(2x^2 - 3)(x^2+1)^{1/2} + 3\ln\left[x + (1+x^2)^{1/2}\right]),$ where $x = \frac{p_F}{m_e c} = 1.2 * 10^{-10} n_e^{1/3}$

dyne cm^{-2}

Fermi momentum

Electron number density

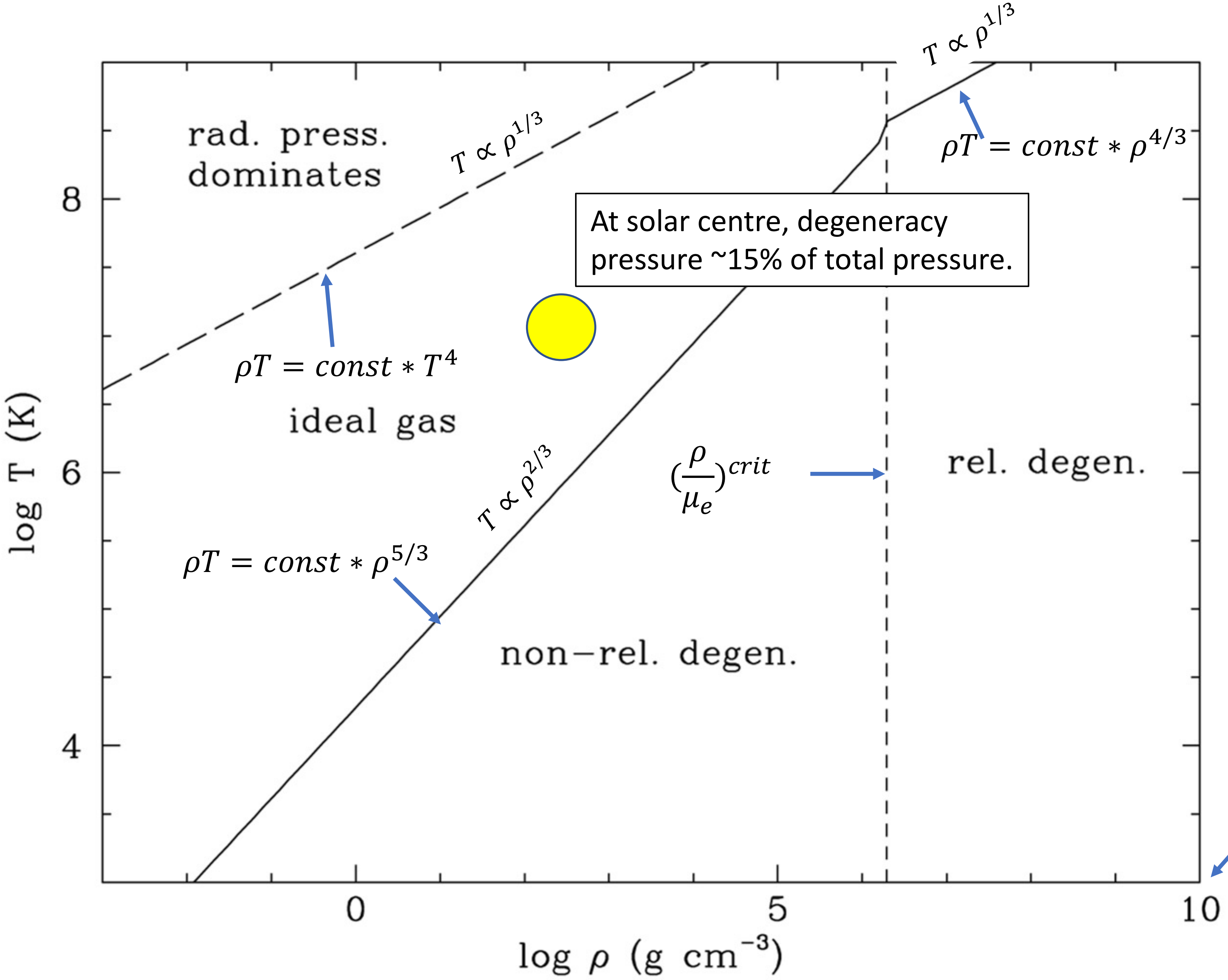
$= 1.0 * 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3}, x \ll 0$ (non-relativistic limit) $U = \frac{3}{2}P, \nabla_{ad} = 0.4$

$= 1.2 * 10^{15}\left(\frac{\rho}{\mu_e}\right)^{4/3}, x \gg \infty$ (ultra-relativistic limit) $U = 3P, \nabla_{ad} = 0.5$

The EOS softens (5/3 to 4/3 exponent) when electrons become relativistic (velocities saturate at $v \rightarrow c$ so cannot increase pressure by higher velocities in addition to higher density). Transition at $x \sim 1 \rightarrow (\rho/\mu_e)^{crit} \sim 10^6$ g cm^{-3}.

Note there is just one "gas pressure" component for each particle type. This limits to nkT when degeneracy effects are negligible, and P_{deg} when they are dominant. See e.g. Kippenhahn Chapter 15 for the general intermediate range.

Equation of state



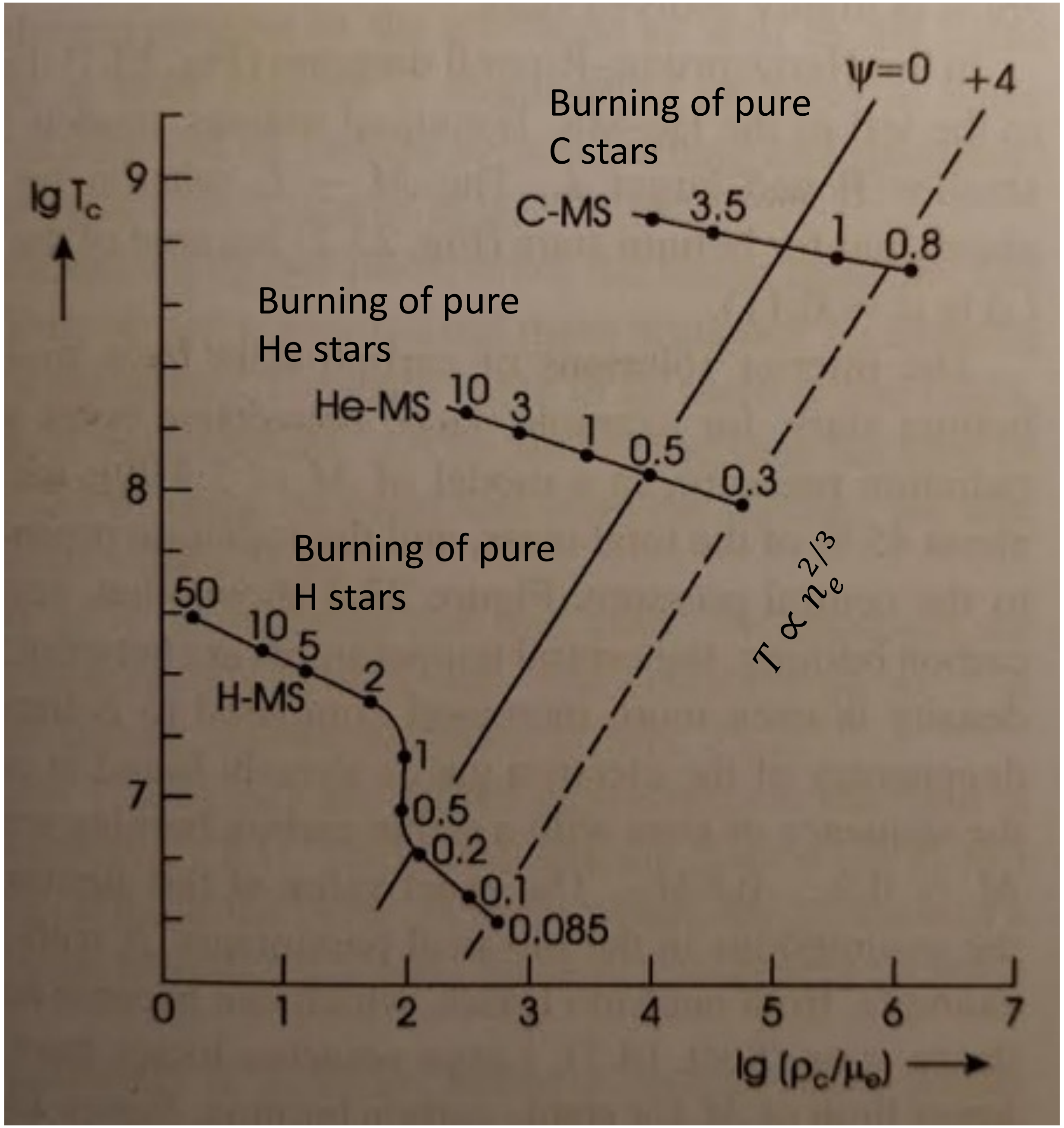
Absolute locations of regime boundaries depend on μ and μ_e, here drawn for particular values of these.

At yet higher densities, ion degeneracy pressure becomes important at

$$\left(\frac{m_{ion}}{m_e}\right)^{3/2} n_e^{crit} = 8 * 10^4$$

$n_e^{crit} A^{3/2} \geq 10^{11}$ g cm^{-3}

When is degeneracy pressure important?



Kippenhahn Fig 23.3

Central densities are higher in **lower-mass stars.**

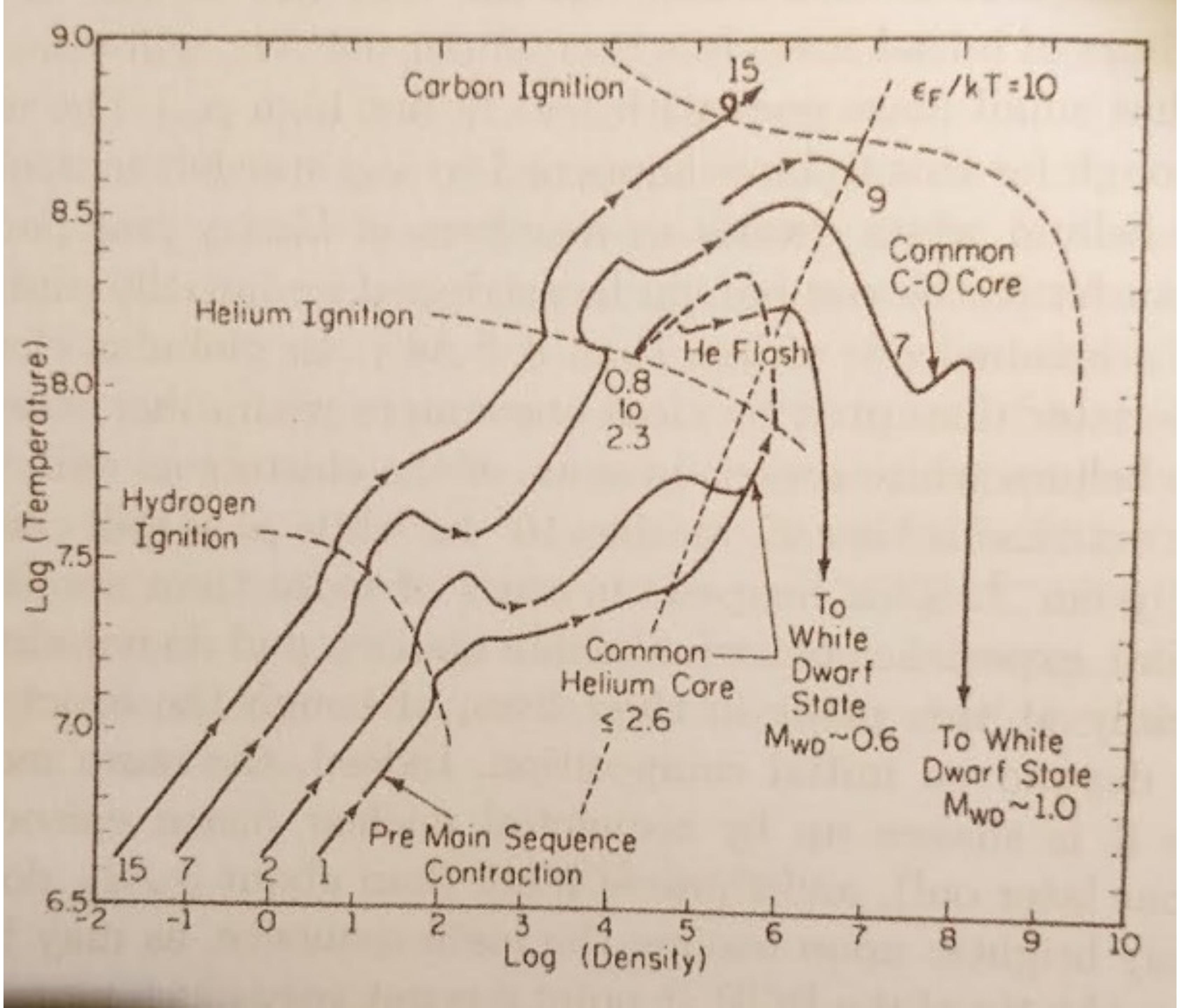
Degeneracy parameter

$$\psi = \psi\left(\frac{n_e}{T^{3/2}}\right)$$

(no analytic form available)

$\psi \to -\infty$ No degeneracy
$\psi \to +\infty$ Strong degeneracy

Degeneracy starts to be important for $\psi \gtrsim 0$.



Degeneracy plays an important role sooner or later for all low and intermediate-mass stars $M_{ZAMS} \lesssim 10\ M_{\odot}$.

More massive stars evolve along tracks avoiding degeneracy regimes.

Hansen Fig 2.12