Acceleration schemes

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June 10 2017

Abstract

Your abstract.

1 Acceleration schemes

1.1 Classic lambda iteration

Classic lambda iteration refers to a scheme where we alternate solutions to the radiation field and to the source function (i.e. gas state).

For a given source function, any method to calculate J_ν can be written in operator form (assuming for now S_ν to be isotropic)

$$J_\nu = \Lambda_\nu[S_\nu] \tag{1}$$

We may also talk about a lambda operation as calculating I and not J, we then add a μ as second subscript

$$I_{\nu\mu} = \Lambda_{\nu\mu}[S_{\nu\mu}] \tag{2}$$

For example, if our method is to do a formal integral solution, $\Lambda_\nu[S_\nu]$ performs integrals in various directions, sampling S_ν from each shell. But it may also be a computer program that implements the (differential) Feautrier scheme (fixed source function variant). Some formal solution methods may correspond to a matrix multiplication with the source function vector, but some involve multi-step procedures that cannot be written in that way.

The work flow is then to initially guess J_ν^0 (for all frequencies and depths), then loop

1. Compute $S_\nu^n = f(J_\nu^n)$ (all frequencies)
2. Compute $J_\nu^{n+1} = \Lambda_\nu[S_\nu^n]$

The idea is that this is faster/more managable than solving the full coupled system, which for a test problem of $S = \varepsilon B + (1-\varepsilon)J$ can be written

$$\mathbf{S} = [\mathbf{I} - (1-\varepsilon)\mathbf{\Lambda}]^{-1}[\varepsilon\mathbf{B}] \tag{3}$$

For example, a problem with 10^4 depth points is too large for direct solution by such inversion, and for complex problems with angle/frequency coupling it is not possible even for small number of depth points (there are limitations from both CPU time and RAM memory).

1.1.1 Convergence issues

Classic Lambda iteration is a perfectly good, and often the best, technique for low and moderate optical depths (e.g. Rutten). For high optical depth, however, convergence of classic Lambda iteration is poor. This is because in a pure iteration, information propagates one optical depth at a time. For an optical depth of τ, photons carry out of order τ^2 scatterings, so as τ gets high the number of iterations needed becomes very large.

This is, however, not the worst part of the story. The more malicious property of the iteration scheme is that correction steps start decreasing with each iteration long before convergence is achieved, giving rise to 'false convergence impression'. As the true solution is (obviously) not known, we cannot easily probe how 'good' the current very slowly changing solution is.

Consider the example of LTE gas with coherent scattering. In the iteration scheme this becomes (assume T is known for now)

$$S_\nu^{n+1} = \varepsilon B_\nu(T) + (1-\varepsilon)\Lambda_\nu[S_\nu^n] \tag{4}$$

Information must propagate to the thermalization depth, which is $\tau_t \sim 1/\sqrt{\varepsilon}$, which requires at least $1/\varepsilon$ iterations.

Significance of ϵ. Does convergence improve of ϵ is large? It does if there is no strong coupling between radiation field and gas state. But if there is such coupling, it is less clear how important ϵ is. **Discuss**

Show figure

A general iterative scheme can be written as

$$x^{n+1} = Fx^n + b' \tag{5}$$

The F matrix is called the **amplification matrix**, and its properties determines convergence properties. One may show

$$e^{n+1} = Fe^n \tag{6}$$

where e is the error vector. Requiring that $||e^{n+1}||/||e^n|| < 1$ requires $||F|| < 1$. Hubeny & Mihalas states 'there are several possible choices for norm, we choose the spectral radius which equals the largest eigenvalue σ". Thus, one gets divergence for $\sigma > 1$, fast convergence for $\sigma << 1$, and slow convergence for $\sigma \sim 1$.

In classic Lambda iteration $F = (1-\varepsilon)\Lambda$. In the optically thick limit $\Lambda \approx 1$ on the diagonal, and if $\varepsilon \ll 1$, $\sigma \sim 1$, predicting bad convergence.

1.2 Accelerated lambda iteration

We can view full coupling and direct inversion (Eq. 3) as one extreme solution approach, and classic lambda iteration as the other extreme, in which the equation system blocks for gas state and RT are completely decoupled in each iteration.

"Accelerated lambda iteration" refers to schemes that are positioned somewhere between these extremes; there is *some* coupling of the source function to the *current* (to be calculated) radiation field. This coupling is in such a way that each iteration is much faster than in the full coupling regime, and each iteration brings a bigger step towards the solution than in classic lambda iteration (because of the partial coupling).

This class of methods are called *approximate lambda iteration*, or equivalently *accelerated lambda iteration*, or *operator perturbation* methods. The first paper introducing them for astrophysics was Cannon 1973, although the philosophy did not take off until the papers by Scharmer 1981,1983 and Olson, Auer and Buchler 1986 (OAB).

The basic idea is to split the lambda operator into two parts:

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*) \tag{7}$$

Here Λ^* is called the approximate Lambda operator. Then, link in the Λ^* part explicitly to act on the *next* S estimate, so change the iteration strategy to

$$S_\nu^{n+1} = \epsilon B_\nu(T) + (1-\epsilon)\Lambda^*[S_\nu^{n+1}] + (1-\epsilon)\left(\Lambda - \Lambda^*\right)[S_\nu^n] \tag{8}$$

We have now made a "semi-implicit" setup where part of the operator acts on the current solution, and part acts on the previous solution. It is clear that *if* we achieve convergence, this is identical to Eq 4, so the solution will be the correct one.

We can rewrite, or 'solve', Eq. 8 for S_ν^{n+1}: (we should combine the equations for each depth, for each of which the Lambda operators define a particular row in a matrix), now also letting Λ specifically be a matrix

$$\begin{aligned} \mathbf{S}_\nu^{n+1} &= [\mathbf{I} - (1-\varepsilon)\mathbf{\Lambda}^*]^{-1}[\varepsilon\mathbf{B} + (1-\varepsilon)\mathbf{\Lambda}[\mathbf{S}_\nu]^n - (1-\varepsilon)\,\mathbf{\Lambda}^*[\mathbf{S}_\nu^n]] & (9)\\ &= [\mathbf{I} - (1-\varepsilon)\mathbf{\Lambda}^*]^{-1}[\mathbf{S}_\nu^{FS,n} - (1-\varepsilon)\mathbf{\Lambda}^*[\mathbf{S}_\nu^n]] & (10)\end{aligned}$$

Solution requires inversion of matrix that is the unity matrix minus a (scaled) $\mathbf{\Lambda}^*$ matrix. If we e.g. choose $\mathbf{\Lambda}^*$ to be diagonal, this will be much faster than to invert the quantity in Eq. 3. Note that we also have to perform 'perturbed formal solutions' $\Lambda^*[S^n]$ in each iteration.

If Λ^* carries enough of the full physics of Λ, we will now get bigger steps towards the true solution in each iteration because of the implicit coupling. To make any gain compared to the fully implicit approach, Eq. 8 must also be solvable with less CPU time and/or memory than Eq 3 (otherwise we might as well do the fully implicit approach). Eq 10 shows that Λ^* needs to be more easily inverted than Λ.

Some insight of the acceleration gained can be seen by writing

$$S^{FS,n} = (1-\epsilon)\Lambda[S^n] + \epsilon B \tag{11}$$

so

$$(S^{n+1} - S^n)_{classic} = S^{FS,n} - S^n \tag{12}$$

With an approximate operator, on the other hand

$$(S^{n+1} - S^n)_{ALI} = [1-(1-\epsilon)\Lambda^*]^{-1}\left[S^{FS,n} - S^n\right] \tag{13}$$

For an heuristic $\Lambda^* = 1$ (e.g. large optical depth $J \approx S$), we get an acceleration of order $1/\epsilon$. (e.g. Eq 5.45 in Rutten) by having coupling. **Discuss: works only for Λ close to 1**

Note that from Eq. 13, iteration may preceed by calculating normal formal solutions, then letting $[1-(1-\epsilon)\Lambda^*]^{-1}$ operate on $S^{FS} - S^n$, and subtract S^n (so no need to let Λ^* operate directly on anything).

1.2.1 Choices of Λ^*

The approximate operator should have the desired properties

- Represent much of the basic physical properties of Λ, to make sure a significant part of the problem gets 'implicitly treated'.
- Be fast and/or require less memory for construction and inversion than Λ

Needless to say, this wishlist opens up an infinite zoo of possible operators, each better or worse for particular applications, and there is a vast literature on this topic.

In the classic literature terms like "diagonal" and "tridiagonal" dominate; the example problem was always a mono-frequency 1D stellar atmosphere, so what varied along the variable array was just the depth point. In a more generalized context, one may talk about the size of the local region for which implicit coupling is expressed, diagonal then corresponds to the local cell, tridiagonal to nearest neighbors etc.

Diagonal/local. In this approach Λ^* contains only contributions by emission in the local cell to create J. In 1D, gray, case that means the diagonal elements of the matrix are the only non-zero ones, and at each depth point there is a simple scalar division. This method was initially presented by Olson, Auer & Buchler 1986, and is also called the OAB method. They use Feautrier variables and 2nd order form of transfer eq (Olson1987), the formal solution is done with the 'Hermite (differential) scheme' of Auer 1976, which is a variant of the Feautrier method. Local operators can be generalized to multi-D (OAB, Rutten).

So what does it actually look like in an example? Note that a diagonal operator does not have to be the exact diagonal of Λ. We are free to choose any approximate formula we want.

- **Escape probability (OAB86)**. $\Lambda^* = 1 - \beta p(\tau)$, where β is a free parameter and $p(\tau)$ is an escape probability.
- **Approximate 2nd order solution (AOB86)**. From

  $$u''(\mu,\tau) = u(\mu,\tau) - S(\tau) \tag{14}$$

  Combine with approximations

  $$u_{i-1} = u_i e^{-\Delta\tau_{i,i-1}/\mu} \tag{15}$$
  $$u_{i+1} = u_i e^{-\Delta\tau_{i,i+1}/\mu} \tag{16}$$

To get (my own derivation, no details in OAB)

$$u_i = S_i \frac{1}{\Delta_1 \Delta_2} \left[-e^{-\Delta_1} - e^{-\Delta_2} + 2 + \Delta_1 \Delta_2 \right]^{-1} \tag{17}$$

Then

$$\Lambda_{ii} = \int \phi_x \int u_i d\mu \tag{18}$$

- **Gray case**, can we use $E_1(0)$ from $J = \int S E_1(t) dt$?

  $$J_\nu(\tau_\nu) = \int S_\nu(t_\nu) E_1(|t - t_\nu|) dt_\nu \tag{19}$$

  Exponential integral is singular at origo..

- **Short-characteristics** (Olson and Kunasz 1987). Appears to be used also in multi-D (Hubeny 2003).

  The formal solution using first-order (linear interpolation) short characteristics is

  $$I(\tau_d, \mu) = I(\tau_{d+1}, \mu) e^{-\Delta\tau_{d+1/2}} + \lambda^+_{d,d} S_d + \lambda^-_{d+1,d} S_{d+1} \quad \mu > 0 \tag{20}$$
  $$I(\tau_d, \mu) = I(\tau_{d-1}, \mu) e^{-\Delta\tau_{d-1/2}} + \lambda^-_{d,d} S_d + \lambda^-_{d,d-1} S_{d-1} \quad \mu < 0 \tag{21}$$
  $$\tag{22}$$

  where

  $$\lambda^+_{d,d} = 1 - \frac{1}{\Delta\tau_{d+1/2}} + \frac{e^{-\Delta\tau_{d+1/2}}}{\Delta\tau_{d+1/2}} \tag{23}$$
  $$\lambda^-_{d,d} = 1 - \frac{1}{\Delta\tau_{d-1/2}} + \frac{e^{-\Delta\tau_{d-1/2}}}{\Delta\tau_{d-1/2}} \tag{24}$$
  $$\tag{25}$$

  where $\Delta\tau_{d-1/2} = (\tau_d - \tau_{d-1})/\mu$. Thus, keeping only the local term, and constructing J from I:

  $$\Lambda^*_\nu = 1 - \frac{1}{2} \int_0^1 \left(\frac{1 - e^{-\Delta\tau_{d-1/2}}}{\Delta\tau_{d-1/2}} + \frac{1 - e^{-\Delta\tau_{d+1/2}}}{\Delta\tau_{d+1/2}} \right) d\mu \tag{26}$$

  Thus, to make the iteration we just multiply $S^{FS} - S^n$ with a scalar correction factor that corresponds to this integral calculation, that for a fixed grid is constant through the iterations.

- **Feautrier scheme** (Rybicki & Hummer 1991). A drawback of the short characteristic method is costly evaluation of exponentials. Rybicki & Hummer showed that one can get the exact diagonal of the Λ matrix in a cheap way by a Feautrier-like approach. For a given frequency-angle point, the solution to u can in general be written

  $$u = \mathbf{T}^{-1} \mathbf{S} \tag{27}$$

  where u is the symmetric average of I, $\mathbf{T}$ is tri-diagonal, and there are N_d equations. From u we get J from

  $$\Lambda = \int \mathbf{T}^{-1} d\mu \tag{28}$$

  Thus we need the diagonal elements of $\mathbf{T}^{-1}$. Rybicki and Hummer showed how these can be found through a recursive sweep algorithmin just of order N_d operations (see e.g. Hubeny and Mihalas page 438 for details).

With diagonal operator, convergence decreases with increased resolution (OAB).

Tridiagonal/one neighbour. This method is less sensitive to grid cell size than the diagonal one. It was introduced by Olson & Kunasz 1987, who suggested to use short characteristics to find the coefficients.

Pentadiagonal and higher. See e.g. Hauschildt 1992, 1994, MacFarline1992. From the literature, it seems the benefits of fewer iterations going to higher bands has not turned out to outweigh the larger computational costs, and Hubeny and Mihalas state that these methods are very rarely used.

Upper tridiagonal. Scharmer 1981 ("core saturation" method). Eddington-Barbier. Trujillo Bueno 1995. Upper diagonal does not extend to multiD (OAB86). Olson & Kunaz 1987, using short characteristics.

- **Core saturation operator**. This method relates to transfer through lines. At high optical depth we know that $J_\nu \approx S_\nu$, so with a one-quadrature formula (Scharmer 1981), we simply get $\Lambda^*_{ii} = 1$. For optically thin case one may instead use the Eddington-Barbier approximation, $J = 1/2 S_\nu(\tau_\nu = \gamma)$.What is counted as optically thick can be controlled with a parameter γ.

- **Scharmer operator**. The Scharmer operator relates to calculation of I, not J. Scharmer set out to find functional forms that have the Eddington-Barbier form ($I_\nu = S_\nu(f(\mu))$

$$\Lambda^*_{\nu\mu} = W^{\pm}_{\nu\mu} S_{\nu\mu}(f_{\nu\mu}(\tau_{\nu\mu})) \tag{29}$$

Eddington-Barbier approximation: Consider the transfer equation

$$\frac{dI_{\nu\mu}}{d\tau_{\nu\mu}} = I_{\nu\mu} - S_\nu \tag{30}$$

Solutions are

$$I^+_{\nu\mu} = \int_\tau^\infty S_\nu(t) e^{-(t-\tau)} dt \tag{31}$$

$$I^-_{\nu\mu} = \int_0^\tau S_\nu(t) e^{-(\tau-t)} dt \tag{32}$$

Now linearise the source function, $S_\nu = a + b\tau_{\nu\mu}$. This gives

$$I^+_{\nu\mu} = a + b + b\tau_{\nu\mu} = 1 \times S_\nu(\tau = \tau_{\nu\mu} + 1) \tag{33}$$

$$I^-_{\nu\mu} = a - b + b\tau - (a-b)e^{-\tau} = \left(1 - e^{-\tau}\right) \times \left(a - b + \frac{b\tau}{1 - e^{-\tau}}\right) \tag{34}$$

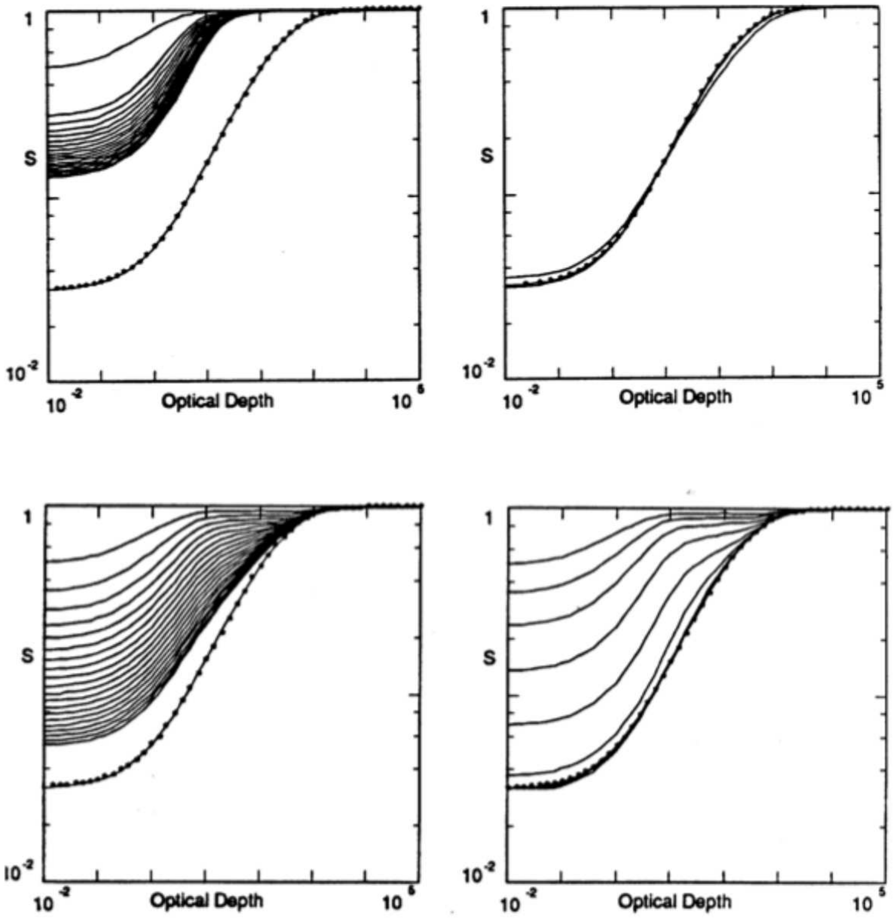
$$= \left(1 - e^{-\tau}\right) S_\nu(\tau = \frac{b\tau}{1 - e^{-\tau}} - 1) \tag{35}$$

One here uses the property that for a linear source function, the exact solution corresponds to S at one particular distance away. Thus, the matrix elements in Λ^* have values $W^+ = 1$ for outgoing beams and $W^- = 1 - e^{-\tau}$ for ingoing beams, and their locations are given by $\tau + 1$ for outgoing and $\frac{b\tau}{1-e^{-\tau}} - 1$ for ingoing. Each row in the Λ^* matrix has exactly one entry. **Does this not become a complex matrix still?**

This operator was implemented together with a complete linearization scheme for level populations in Scharmer and Carlsson 1985. An advantage of the Scharmer operator is that it allows for anisotropy in S (Rutten). However it does not generalize to multiD.

1.2.2 Extension from plane-parallell to spherical geometry

Hamann 1985, 1986, 1987, Hempe & Schoenberg 1986, Schoenberg and Hempe 1986, Puls & Herrero: spherical case (sometimes velocity gradient). Diagonal operators.
Hillier 1990 (CMFGEN) : spherical, velocity gradient, multilevel. Tridiagonal, pentadiagonal operator, but note that there are some subtle differences between Hilliers method and classic ALI, which leads Hubeny and Mihalas to classify them as "approximate Newton Raphson" rather than ALI.



1.2.3 Extension to NLTE multilevel

The more generic multi-level problem can be stated

$$J_\nu = \Lambda[S_\nu] \tag{36}$$
$$\mathbf{n} = f_1(J) \tag{37}$$
$$S_\nu = f_2(\mathbf{n}) \tag{38}$$

These equations may also be supplemented by an energy equation (radiative equilibrium for stellar atm.) and hydrodynamic equations (hydrostatic equilibrium).

The level populations are normally considered separate for each species I (all ion. stages), and when there is coupling (e.g. with charge transfer) one can typically use the last iterate for the partner populations (Hubeny & Mihalas Ch 18). Treating also n_e as fixed, the steady state rate equations form a linear system

$$\mathbf{A}_I \mathbf{n}_I = \mathbf{b}_I \tag{39}$$

One of these equations has to be replaced by a number conservation equation. This choice is arbitrary. The set of equation blocks for each species I is complemented with a charge neutrality equations, matching n_e with the sum of all ions.

Eqs. 38 provide a large set of coupled, non-linear equations (e.g. in rate equations level populations are multiplied with J). "Complete linearization" is the term RT theorists use for solving this full system by Newton-Raphson. All variables can be put together into a "state vector" in each cell

$$\psi_d = \{J_1, .., J_{NF}, T, n_e, n_1, .., n_{NL}\} \tag{40}$$

of size $NF + NL + NC$, where NC is number of constraints (1-3).

All partial derivatives for all equations are computed, which forms the Jacobi matrix, and iteration preceeds by

$$\Delta\psi = \mathbf{J}^{-1}\mathbf{e} \tag{41}$$

where $\mathbf{e}$ is the current error matrix. In practise we can arrange the equations so to obtain a block tridiagonal Jacobian, so inversion takes $N_d \left(N_{NF} + N_{NL}\right)^3$ operations (using a Feautrier-like technique).

In the old literature problems where $NF \sim 100$, $NL \sim 10$ were treated, which meant that equation system of size ~ 1000 were inverted. However, in modern applications $NF \sim NL \sim 10^5$, and simultaneous solution of the whole equation system (e.g. complete linearization) is ruled out. **Hubeny & Mihalas : This may perhaps be done on large parallell systems..discuss**

In the general case its impossible to explicitly link J back into the RT equation; S depends on the solution of a set of non-linear equations involving J. But with ALI we may use Λ^* for simple relations between J and S that can be worked in analytically or by compact matrix generation code

Rutten describes Auer & Mihalas 1969 as first paper outlining complete linearization, i.e. coupled solutions of statistical equilibrium and RT equations. Here level population correction are replaced by sums over radiation field corrections, and these are solved for using the transfer equation (Feautrier scheme in Auer and Mihalas). No real explanations in Rutten.

Scharmer & Carlsson 1985, on the other hand, use an approximate transport equation when iterating the level populations. Note that the final answer is always the output of the 'classic' lambda iteration, which can be seen as doing the approximate step plus the correction step.

The problem is really fully specified by all level populations at all depths. Thus, if we can do ALI in the J domain, it must be equivalent to doing it in the n domain, where some degree of neighbour implicit coupling in each iteration, benefits convergence. This must be alternated with the 'exact' solution using the last guess for neighbors. Thus, one must do something clever/coupled, alternated with standard, local, (linear) equation system solutions. Both MULTI and CMFGEN do this (but the CMFGEN papers dont explicitly state this second correction, or lambda iteration, step).

1.2.4 Extension to multi-D

Hubeny 2003 states "diagonal operator is the only practical choice for multiD cases...tridiagonal and higher methods extend with difficulty to multi-level problems". No references for these statements. OAB86 make similar statements.

1.3 External accelerations

1.3.1 Ng acceleration

In Ng acceleration (Ng 1974), we aim to minimize the difference between a new estimate and the next iteration based on that estimate (at convergence there would be no difference). If we have a solution x^n for iteration n, let an extrapolated solution $\hat{x}^n$ be some linear combination of the last 3 solutions (one may also use more/less but this is the common choice)

$$\tilde{x}^n = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} \tag{42}$$

under the constraint $c_0 + c_1 + c_1 = 1$. Now lets say we want to minimize $F[\tilde{x}^n] - \tilde{x}^n$ (i.e. the next correction to our extrapolation), or rather the norm $|F[\tilde{x}^n] - \tilde{x}^n|$. This is a least-squares problem (e.g. Hubeny and Mihalas), fitting two values c_1 and c_2 to minimize a sum of residuals. We are basically saying, there is already a lot of converged or quasi-converged information in our current estimate x^n, but scalings are off and we want to correct those using c_1 and c_2.

Now, if F is linear, $F[\tilde{x}^n] = c_0 F[x^n] + c_1 F[x^{n-1}] + c_2 F[x^{n-2}] = c_0 x^{n+1} + c_1 x^n + c_2 x^{n-1}$. Thus with $x^{n+1}, x^n, x^{n-1}, x^{n-2}$ known, its an algebraic sum involving c_0, c_1, c_2, easily calculable.

The solution is (Hubeny & Mihalas, see also OAB 1986)

$$c_1 = (\delta_{01}\delta_{22} - \delta_{02}\delta_{21})/(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) \tag{43}$$

$$c_2 = (\delta_{02}\delta_{11} - \delta_{01}\delta_{21})/(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) \tag{44}$$

where

$$\delta_{ij} = \left(\Delta x^n - \Delta x^{n-i}\right) \cdot \left(\Delta x^n - \Delta x^{n-j}\right) \tag{45}$$

and

$$\Delta x^n = x^n - x^{n-1} \tag{46}$$

Note that we need 4 consecutive solutions to make one acceleration. In practise one minimizes a residual that uses weights W_i that can be chosen by various arguments (e.g. OAB86 uses $W_i = 1/x_i$):

$$\Omega = \sum_i \left[\tilde{x}_i - F(\tilde{x}_i)\right] W_i \tag{47}$$

One may optimize over c coefficients locally (let them vary for each depth), or globally (one and the same for each depth).

Ng accelaration can be done earliest in the 4th iteration, although one normally waits somewhat longer as initial iterates are still far from convergence.

Ng acceleration is particularly useful in combination with ALI methods that are only 'moderately' fast in convergence on their own, such as local operators (OAB86).

1.3.2 ORTHOMIN

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1.3.3 Chebyshev acceleration

See Olson 1987. Hubeny & Mihalas : Not clear if can be used as a general RT method.

1.3.4 Broyden method

Instead of using Jacobian matrix, use Broyden matrix, which correspond to secant method in 1D. Each inverse Broyden matrix can be calculated from the previous inverse without new inversion.

1.3.5 Kantorovich method

This method is simply to hold the Jacobian constant once one approaches convergence ("keep tangent fixed"). Method has been shown to be surprisingly robust in RT context. One performs a few normal NR iterations and then swicthes to constant Jacobian. A refreshment may be done at some point.

1.3.6 Solution guiding

Any iterative scheme needs a good starting guess (even complete linearization), and sometimes active guidance also in later steps. For example, one may "switch on" more and more detailed physics in a a series of iterations such as

LTE-gray → LTE → NLTE-continua → NLTE-few lines → NLTE-many lines This may bring about a solution in fewer steps than doing the full NLTE-lines calculations from scratch.

1.3.7 Successive overrelaxation

E.g. Trujillo Bueno 1995.

1.3.8 Problem simplification

To reduce the number of frequencies, levels and depth points that are implicitly coupled, one may consider

- Implicit coupling only for some selected (judged important) frequencies and levels.
- Divide levels into *superlevels*, where LTE is assumed within each superlevel. A related idea is *level grouping*, where corrections within a group of lines is constant.
- Reduce N_D. Most algorithms are linear in N_D so no big gain. Note that AMR methods have not yet been used in RT codes, and this may be one of the most important remaining improvements (Hubeny & Mihalas).

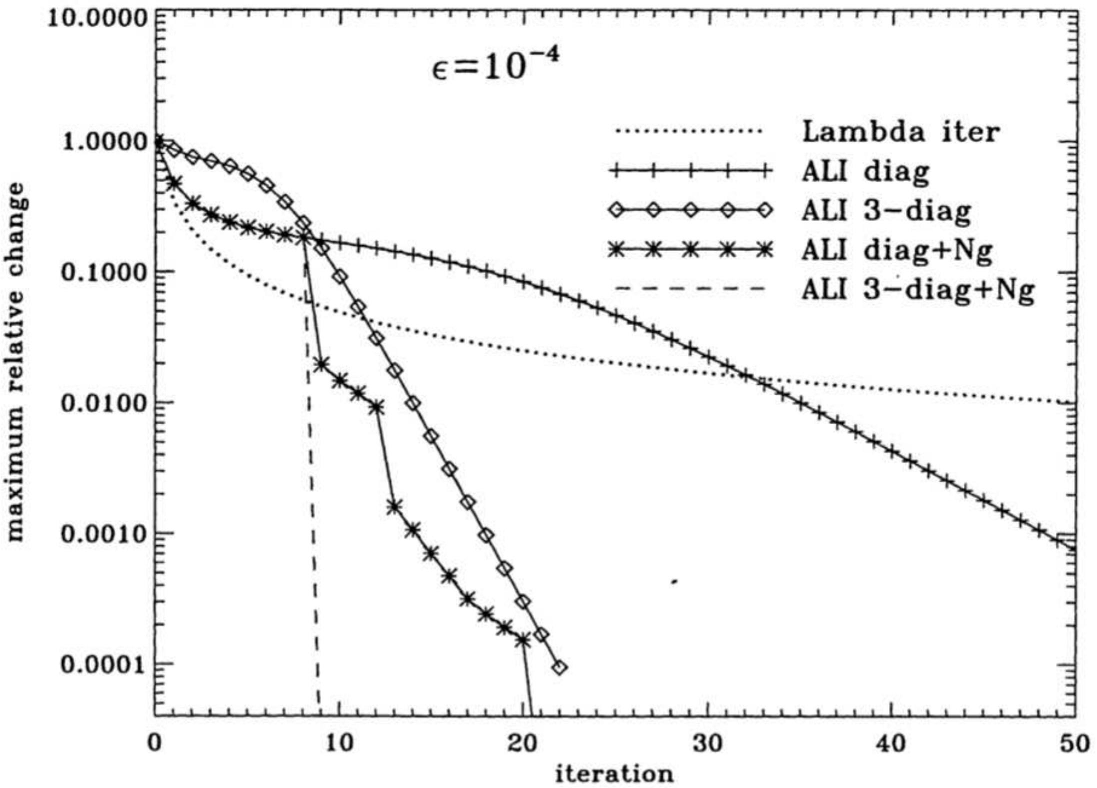

Figure 1: Comparison of iteration schemes with and without Ng acceleration, from Hubeny 2003.