Astrophysical Gasdynamics Lecture Notes

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1 Introduction

This course is about the dynamical behaviour of gas in astrophysical systems. It is one of the fundamental pieces of physics needed in astrophysics, as at least 90% of all (baryonic) matter in the Universe can be described as a gas. The other fundamental physical processes governing astrophysical systems are gravity and the interaction between radiation and matter. Most astrophysical systems can therefore be described as a gas interacting with radiation under the influence of gravity. The major constituents of the Universe consist all of gases. In order of increasing size scale they are

- Stars: not solids, but self-gravitating gaseous bodies.
- Interstellar medium (ISM): gas inside galaxies consisting of a mix of cold molecular clouds, cool HI clouds, warm ionized gas, hot ($> 10^6$ K) gas (multiphase ISM). In the Milky Way the ISM makes up $\sim 20\%$ of the mass.
- Intergalactic gas: low density gas between galaxies. Inside clusters of galaxies (intracluster medium or ICM): dominant part of cluster mass (hot X-ray emitting gas); Outside clusters (intergalactic medium or IGM): substantial part of the mass in the Universe.

What is a gas? The practical answer is that it is a collection of microscopic particles whose properties can be described by macroscopic *continuum* quantities (density, velocity, pressure, temperature, entropy, etc.). In this sense gas is a ***fluid***, just as liquids are fluids. However, unlike liquids, gases are compressible (the density can vary). For gas on Earth the compressability is typically low (and often neglected), but for astrophysical gases it is high. The reason for the compressability of gases is that the gas particles are relatively far apart compared to the particles in a liquid.
Gases and liquids are not the only systems that can be described as a continuum. Other examples are

- Stellar systems consisting of stars
- Dust systems (e.g. Saturn's rings) consisting of dust particles
- Radiation consisting of photons
- Traffic systems consisting of cars

In all of these systems the particles interact with each other, but not continuously, that is the interactions can be described as 'collisions'. In some systems these collisions are rather rare (stellar systems, dust systems), and they are referred to collisionless. In normal gases collisions are important, and such collisional gases are the subject of this course.

There are a number of other terminologies being used for gas dynamics. It is good to be aware of this in order to prevent confusion. Gas dynamics is often referred to as *hydrodynamics*, *fluid dynamics*, or *aerodynamics*. The latter term is only used in Earth-based applications. 'Hydrodynamics' is officially defined to be the dynamics of liquids, but in astrophysics it is often used to mean dynamics of gases. The term 'Fluid dynamics' covers both liquids and gases, but often it is used for gases alone (as in the title of the course book "Principles of Astrophysical Fluid Dynamics").

When the gas consists of ions and electrons, the particles can interact with any magnetic and/or electric fields present, as well as with each other. This is situation is most generally described by *plasma physics*, or its less general variety *magnetohydrodynamics* (MHD). In this course we will not consider magnetic fields, but it is good to remember that magnetic fields are present in most astrophysical systems and a more complete description should take them into account.

The study of fluid dynamics has a considerable history with a lot of activity in the 18th and 19th centuries. We will encounter names such as Bernoulli, Euler, Lagrange, Navier, Stokes, von Helmholtz, Kelvin. In the 20th century the theoretical interest diminished as focus shifted to relativity and quantum physics. However, this period also saw the development of many new applications of fluid dynamics: flight (sub– and supersonic), atomic bomb explosions, re-entry of space vehicles. Also plasma physics and magnetohydrodynamics was first developed in the 20th century, a key figure being the Swede Hannes Alfvén. Important applications of magnetohydrodynamics have been the development of nuclear fusion reactors, as well as studies of the Earth's ionosphere. Another remarkable development has been the possibility to study solutions of the flow equations using numerical techniques. This enabled the study of flows in a lot more detail than in the laboratory. Even though gas and fluid dynamics are no longer at the forefront of theoretical physics, this does not mean that every aspect is fully understood. Our theoretical understanding of processes such as turbulence and convection are only limited and at the fundamental mathematical level it has not been proven that given an initial condition in $\mathbb{R}^3$ there are smooth solutions for the gas dynamic equations for all future times. This is one of the seven *Millenium Prize Problems* as chosen by the Clay Mathematics Institute of Cambridge, Massachusetts in 2000.

Read pages 1 – 4 (up to Section 1.2) of Clarke & Carswell.

1.1 Properties of gases

Gases consists of particles which interact with each other through collisions. Through these collisions, the particles act as a collective, and a continuum description is possible. One can thus say that there is a microscopic view of a gas (particles) and a macroscopic view (continuum). The typical quantities used to characterize a gas are the macroscopic ones: density, pressure, temperature, velocity, entropy, etc. These are the types of quantities you have worked with in thermodynamics.
To be able to describe a collection as particles as a continuum fluid, it needs to fulfill certain criteria. First of all one needs to be able to define regions called *fluid elements* which

- are large enough to construct a meaningful average macroscopic quantity from the microscopic properties of the individual particles. This means that if the number density of particles is n, the size of the region should satisfy

$$n\ell_{\rm region}^3 \gg 1\,. \tag{1}$$

  In other words the region should contain many particles.

- are small enough that we can ignore variations in macroscopic quantities across them. This means that the size of the region $\ell_{\rm region}$ is much smaller than a scale length for any variations in quantity q. Or

$$\ell_{\rm region} \ll \ell_{\rm scale} \sim \frac{q}{|\boldsymbol{\nabla} q|}\,. \tag{2}$$

In addition, a collisional fluid needs to fulfill the conditions that the interaction scale or mean free path between collisions of particles λ, is much less than the size of the fluid element:

$$\lambda \ll \ell_{\rm region}\,. \tag{3}$$

The cases we will consider in this course are all collisional fluids. The effect of the collisional fluid condition is that the particles within a fluid element collide so often with each other that they have reached a unique equilibrium distribution in their velocities. This allows us to define a *local* velocity, temperature and pressure. An example of a non-collisional fluid problem would be collision of two stellar systems (galaxies). Since the stellar densities are low, the individual stars do

not interact with each other during the collision, but only react to the collective gravitational potential of all the stars together. The result is that the galaxies will fly through each other. In a given region of space there will thus be stars moving in one direction (belonging to galaxy 1) and stars moving in another direction (belonging to galaxy 2). Averaging the stellar velocities in this region is not useful, since you will find, for example in a head-on collision, a very low average velocity, whereas there are large systematic motions in the region. In order to describe this region you have to know from which galaxy the stars came, or in other words you need to know their initial conditions.

As an aside it can mentioned that we see concrete evidence that stars do act this way. Even today one can find 'stellar streams' in the Milky Way which are remnants of collisions which happened long ago.

Gas particles have a much higher space density than stars and in a collision between galaxies any gas contained in the galaxies will therefore react very differently from the stars. In our simplified example of a head-on collision, the gas will attain a very low average velocity, but with a large random motion component of the gas particles, or in other words a high gas temperature. The many collisions that have happened between the gas particles have also erased any information on where they originally came from. The motions of the gas particles are now characterized by the local temperature T and not by their initial conditions.

This difference between the collisional behaviour of gas and collisionless behaviour of other materials has been used as a proof of the existence of dark matter. In the collision of two clusters of galaxies, the collisionless dark matter particles moved through each other, whereas the gas particles in the two clusters collided and remained stuck in the middle, reaching high temperatures. Mapping the dark matter through gravitational lensing and the gas through x-ray observations, this effect was seen in the so-called Bullet cluster (Fig. 1).

The equilibrium distribution achieved by the particles' velocities corresponds to a state of maximum entropy. For a stationary collection of particles of identical mass m this distribution is the *Maxwell-Boltzmann distribution function*

$$n_{\mathrm{MB}}(v)\mathrm{d}v = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left[-\frac{mv^2}{2k_{\mathrm{B}}T}\right]4\pi v^2\mathrm{d}v\,, \tag{4}$$

where n is the number density of particles, v is the particle speed (absolute velocity) and T is the temperature. The constant k_{B} is the Boltzmann constant, connecting energy and temperature. For a Maxwell–Boltzmann distribution the

Figure 1: The Bullet cluster (1E 0657-56). The purple colour indicates the x-ray emission from the gas (observed with the Chandra x-ray telescope), whereas the blue colour indicates the distribution of gravitational matter (mostly dark matter), mapped using gravitational lensing.

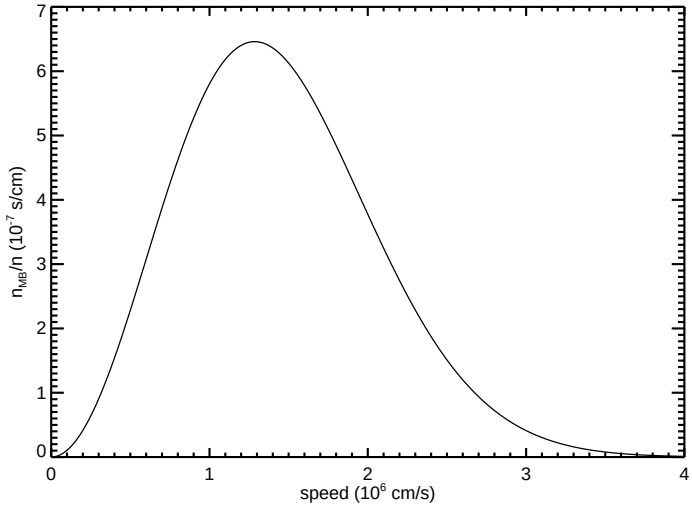

Figure 2: The Maxwell-Boltzmann distribution of absolute velocities for a hydrogen gas of temperature 10^4 K. The most probable velocity is 1.20×10^6 cm s^{-1}. The area under the curve between two speeds is equal to the fraction of gas particles in that range of speeds.

most probable velocity of a particle is

$$v_{\rm mp} = \sqrt{\frac{2k_{\rm B}T}{m}} \tag{5}$$

implying that the most probably kinetic energy of a particle is $k_{\rm B}T$. However, some particles have much lower and some much higher velocities and energies (see Fig. 2).

As we will see, the fact that the particles follow this particular distribution, allows us to find a relation between the density, temperature and pressure (p) of the particles, the so-called Equation of State (EOS). For the Maxwell-Boltzmann distribution, this EOS is

$$p = nk_{\rm B}T\,, \tag{6}$$

also known as the ideal gas law.

1.2 Mathematical concepts

As we will see the basic equations of gas dynamics are partial differential equations involving three-dimensional vectors. It is therefore useful to review the mathematics of 3D vector operations. A separate document called *VECTOR CALCULUS: USEFUL STUFF* gives an overview of this.

Read through *VECTOR CALCULUS: USEFUL STUFF*.

Make sure you understand the use of the various vector operations. Which operations turn vectors into scalars? Which turn scalars into vectors? Which turn vectors into vectors? Pay special attention to the operator $\mathbf{F} \cdot \boldsymbol{\nabla}$ which may be new to you, but which we will encounter often. It is also known as the *comoving derivative*; $(\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{A}$ is the derivative of vector $\mathbf{A}$ along vector $\mathbf{F}$. Also make sure to familiarize yourself with the *suffix notation* as we will frequently encounter it. In addition to what is contained in that document, please note the following:

- We will follow the usual convention that a **boldface** symbol represents a vector. On the board I may use the alternative notation $\vec{a}$. Tensors may be represented using capitals in a sans serif font, A, or by $\vec{\vec{A}}$ on the board.
- The book uses the symbol $\wedge$ where we use $\times$. So the outer product of two vectors in the book is written as $\mathbf{a} \wedge \mathbf{b}$, but we will use the more usual form $\mathbf{a} \times \mathbf{b}$. The same goes for the curl operator: where the book uses $\boldsymbol{\nabla} \wedge \mathbf{a}$ we use $\boldsymbol{\nabla} \times \mathbf{a}$.
- The *suffix notation* is sometimes known as the (Einstein) summation convention (Clarke & Carswell use this terminology). We will use latin symbols such as i and j, implying summation from 1 to 3. In relativity greek symbols are used (α, β) to imply summation from 0 to 3, where 0 represents the temporal dimension.
- We will also encounter the use of curvilinear coordinates (cylindrical and spherical coordinates). Appendix A.2 of Clarke & Carswell summarizes in very general terms how the vector operators change when one is using curvilinear coordinates. The hand-out *DIFFERENTIAL OPERATORS IN CURVILINEAR COORDINATES* does this more explicitly for cylindrical polar and spherical polar coordinates. You can consult this document once we encounter the curvilinear coordinates.

- We will also encounter the so-called dyadic operator which turns two vectors into a two-dimensional tensor:

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} . \tag{7}$$

2 Crowd Control

2.1 Introduction

Any gas consists of particles (atoms, ions), and considering the properties of these particles is called the microscopic view. Properties of particles are

1. Number of particles N
2. Position of a particle $\mathbf{x}$
3. Velocity of a particle $\mathbf{v}$
4. Other intrinsic particle properties (mass, charge, etc.).

When we measure gas properties in a lab, we instead are considering measurable, collective properties of the particles. What macroscopic quantities define the state of a gas?

1. How much there is, so the mass density ρ, or the number density n.
2. How it moves, so the gas velocity $\mathbf{u}$, or momentum density $\rho\mathbf{u}$ (systemic or average motion of the particles).
3. The pressure p it exerts external objects (e.g. a wall or a container) p, which is as we know from thermodynamics connected to the temperature T, or internal/thermal energy $\rho\mathcal{E}$ or the gas through the Equation of State (EOS):

$$p = nk_{\mathrm{B}}T \tag{8}$$
$$p = (\gamma - 1)\rho\mathcal{E}\,, \tag{9}$$

   where γ is the so-called adiabatic index, connected to the number of degrees of freedom of the particles; it is 5/3 for a monatomic gas.

4. Other quantities, such as composition, magnetic field, etc..

The minimum set is the first three (ρ, $\mathbf{u}$, p), so in order to study the dynamics of a gas, one need to derive a set of equations describing the time evolution of these quantities. In other words we need to derive mathematical expressions for $\partial\rho/\partial t$, $\partial\rho\mathbf{u}/\partial t$ and $\partial p/\partial t$.

2.2 Ways to derive the fluid equations: conservation principles

To arrive at these equations there are different ways. One is to use conservation principles (for mass, momentum, energy) and concepts from thermodynamics. This derivation relies partly on the *divergence theorem* or *Gauss' theorem* from fundamental calculus.
Gauss' theorem states that if one has a vector $\mathbf{U}$ then for a volume V enclosed by a surface S, the following equality holds

$$\oint_S \mathbf{U} \cdot \mathrm{d}\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{U} \mathrm{d}V \,. \tag{10}$$

This is a purely mathematical theorem, that has a strong connection to physics, especially gas dynamics.
Now if we have quantity Q of something in volume V, let us say mass M, then we define the mass density ρ through

$$M = \int_V \rho \mathrm{d}V \,, \tag{11}$$

the mass in volume V is the volume integral over the mass density. This mass may be changing because the gas is moving with a velocity $\mathbf{u}$. Obviously the change in mass must be the difference between the amount of material that flows in and out of V. If we now divide the surface S of V into small sections $\mathrm{d}\mathbf{S}$ then the *mass flux* across such a section will be $\rho\mathbf{u} \cdot \mathrm{d}\mathbf{S}$, since only the component of $\mathbf{u}$ into or out of V matters (that is, the component of $\mathbf{u}$ perpendicular to $\mathrm{d}\mathbf{S}$). If $\rho\mathbf{u} \cdot \mathrm{d}\mathbf{S}$ is positive, material is flowing out of V.
Thus the rate by which is the mass is changing inside V is given by the integral over all small sections $\mathrm{d}\mathbf{S}$ of the entire surface S:

$$\frac{\partial M}{\partial t} = -\oint_S \rho\mathbf{u} \cdot \mathrm{d}\mathbf{S} \tag{12}$$

where the minus sign makes sure that inflow will increase the mass. Using Gauss' theorem this becomes

$$\frac{\partial}{\partial t} \int_V \rho \mathrm{d}V = \int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) \mathrm{d}V \,, \tag{13}$$

which being true for all volumes, becomes

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0 \,, \tag{14}$$

describing the time evolution of the mass density ρ. This equation is known as the **continuity equation**.
Obviously, this equation holds for any other quantity which can only change because of being transported by the flow. An example is the number of particles n or partial densities (such as the density of helium in a gas consisting of a mix of elements). Below we will see another example of the continuity equation in the context of density in what is known as six-dimensional *phase space*.
Clarke & Carswell describe in somewhat more detail how conservation principles lead to the fluid equations.

> Read Chapter 2 of Clarke & Carswell to see how the fluid equation can be derived from conservation principles employing the divergence theorem. Please note that we will treat Lagrangian versus Eulerian points of view below.

2.3 Ways to derive the fluid equations: statistical mechancis

The other way to derive the fluid equations is to use the tools of statistical mechanics, i.e. considering a gas as a collection of particles. Because the latter gives us a better insight in when the equation of gas dynamics apply, and the fact that gas consists of particles is conceptually important in astrophysical contexts, these lecture notes provide a sketch of how the derivation is done in statistical mechanics. If you are eager to learn more, a more detailed description of how to derive the gasdynamic equations using statistical mechanics can for example be found in the textbook *The Physics of Fluids and Plasma* by A. R. Choudhuri.

2.3.1 Distribution function

Consider a collection of N gas particles of equal mass m. This is the microscopic view of a gas. Each particle has a position $\mathbf{x}$ and a velocity $\mathbf{v}$. We can thus put it in a 6-dimensional *phase space* (which we call μ) of position and velocity, and count the number of particles in the 6-dimensional volume $(\mathbf{x} - \delta\mathbf{x}/2 : \mathbf{x} + \delta\mathbf{x}/2, \mathbf{v} - \delta\mathbf{v}/2 : \mathbf{v} + \delta\mathbf{v}/2)$. Doing this gives us the *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$:

$$N(\mathbf{x} - \delta\mathbf{x}/2 : \mathbf{x} + \delta\mathbf{x}/2, \mathbf{v} - \delta\mathbf{v}/2 : \mathbf{v} + \delta\mathbf{v}/2), t) = f(\mathbf{x}, \mathbf{v}, t)\delta\mathbf{x}\delta\mathbf{v}\,. \quad (15)$$

The distribution function is effectively a particle number density in the six-dimensional μ-space.

You may have encountered distribution functions before. Famous distribution functions are the Bose-Einstein and Fermi-Dirac distribution functions, for bosons and fermions respectively. Since we are not dealing with quantum effects, we can use that other famous distribution function, the *Maxwell-Boltzmann distribution function*

$$f_{\mathrm{MB}}(\mathbf{v}) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left[-\frac{m(\mathbf{v}-\mathbf{u})^2}{2k_{\mathrm{B}}T}\right], \tag{16}$$

which gives the distribution of the particles over velocities in an equilibrium system at temperature T. Note that Eq. 16 gives the form for the distribution of the three-dimensional velocity $\mathbf{v}$, whereas the Eq. 4 gave it for the absolute velocity v. The Maxwell-Boltzmann distribution function does not depend on position because the system is at equilibrium and the particles have the same velocity distribution at all positions. The microscopic quantities are the particle mass m and velocity $\mathbf{v}$, and the macroscopic quantities are the temperature T, the number density of particles n, and the gas velocity $\mathbf{u}$ (mean velocity of the particles). k_{B} is the Boltzmann constant, connecting energy and temperature.
In general, the macroscopic quantities that we are after (gas density, velocity, and energy) can be derived from the distribution function f by integrating over the particle velocities:

$$\rho(\mathbf{x}) = \int m f(\mathbf{x},\mathbf{v}) d\mathbf{v} \tag{17}$$

$$\mathbf{u}(\mathbf{x}) = \int \mathbf{v} f(\mathbf{x},\mathbf{v}) d\mathbf{v} \Big/ \int f(\mathbf{x},\mathbf{v}) d\mathbf{v} \tag{18}$$

$$E(\mathbf{x}) = \int \frac{1}{2} m v^2 f(\mathbf{x},\mathbf{v}) d\mathbf{v} \Big/ \int f(\mathbf{x},\mathbf{v}) d\mathbf{v}\,. \tag{19}$$

These are actually the 0^{th}, 1^{st} and 2^{nd} moments of the distribution function over the velocities.
To find the time evolution of these quantities we need to find the evolution of f: $\mathrm{d}f/\mathrm{d}t$, which we will address in the following sections.

2.3.2 Collisionless systems

What is $\mathrm{d}f/\mathrm{d}t$? Since f is a function of t, $\mathbf{x}$ and $\mathbf{v}$, we can start by considering the partial derivatives.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{v}} \tag{20}$$

Let us now consider the case of no collisions, a so-called collisionless system, where the only force working on the particles is a possible outside one. From classical mechanics we know that such a system can be described by a Hamiltonian $H(\mathbf{x}, \mathbf{v}, t)$ equal to $H = v^2/2 + \phi(\mathbf{x})$, where ϕ is the potential describing the outside forces[1]. The equations of motion follow from

$$\begin{aligned} \dot{\mathbf{v}} &= \mathbf{a} = -\boldsymbol{\nabla} H \\ \dot{\mathbf{x}} &= \mathbf{v} = \boldsymbol{\nabla}_v H \,, \end{aligned} \tag{21}$$

where $\boldsymbol{\nabla}_v$ is the gradient in the velocity coordinate: $(\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$. The evolution of f in this case is particularly simple, given by $\mathrm{d}f/\mathrm{d}t = 0$. This can be shown by realizing that f is a density in phase-space, and hence must obey the equation of continuity (Eq. 14, derived from the principle of mass conservation: no particles are created or destroyed, they just move around in phase space). This means that

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla} \cdot f \mathbf{u}_f = 0 \,, \tag{22}$$

where $\mathbf{u}_f$ is the 'velocity' for the density f. In μ-space this 'velocity' is the six-dimensional vector $(\mathbf{v}, \mathbf{a})$, so the continuity equation can be written as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (f\mathbf{v}) + \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a}) = 0 \,. \tag{23}$$

This can be rewritten as

$$\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} + f \left(\boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}_v \cdot \mathbf{a} \right) = 0 \,. \tag{24}$$

From the Hamilton relations (Eq. 21) it follows that

$$\boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}_v \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}_v H) - \boldsymbol{\nabla}_v \cdot (\boldsymbol{\nabla} H) = 0 \,. \tag{25}$$

So, the evolution of f can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \,, \tag{26}$$

an equation known as the ***Collisionless Boltzmann Equation***. This equation can be used for all kinds of systems consisting of particles for which collisions are

[1] If the forces on a particle depend on neighbouring particles, for example due to collisions, we cannot find an H which depends only on $\mathbf{x}$ and $\mathbf{v}$.

unimportant, for example low density stellar systems such as galaxies (but not the cores of the much denser globular clusters).
If we consider a volume $\delta\mathbf{x}\delta\mathbf{v}$ in phase space, then it will contain $N = f\delta\mathbf{x}\delta\mathbf{v}$ particles (Eq. 15). Following these particles, they will at a later time be contained in a volume $\delta\mathbf{x}'\delta\mathbf{v}'$. However, for a collisionless system, f will not have changed in this volume (since $\mathrm{d}f/\mathrm{d}t = 0$), and since particle number N is conserved, this implies that $\delta\mathbf{x}\delta\mathbf{v} = \delta\mathbf{x}'\delta\mathbf{v}'$, i.e. the shape of the volume can change, but not its total value. This is generally true for a system which can be described by a Hamiltonian and is known as Liouville's theorem.

2.3.3 Collisions

For a typical gas, collisions *are* important, so Eq. 26 has to be modified. An important parameters when considering collisions is the mean free path between collisions, given by

$$\lambda^{-1} = \sqrt{2}\pi a^2 n\,, \tag{27}$$

for spherical particles of number density n and particle size a. As we saw in Sect. 1.1, Eq. 3, for collisions to be important we require that $\lambda \ll L$, the size of our system. To be able to consider collisions to be a pertubation (rather than a permanent condition of the collection of particles), they should be rather rare, so we need $\lambda \gg a$. In this case we speak of a 'dilute gas' as the particles can travel many times their own size before colliding with another particle. This is the usual case in astrophysics.
For this dilute case, the effects of collisions can be added as a perturbation to the case of no collisions. Most of the time particles move around in phase space following continuous trajectories set by their velocity and acceleration, but every now and then they jump to another trajectory due to a collision (see Fig. 3). The collisions are thus short range, and also binary (for a dilute gas we do not need to consider collisions of three or more particles). We therefore rewrite Eq. 26 as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = C(f)\,, \tag{28}$$

where $C(f)$ depends on the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ and describes the effects of collisions. If we can find a good mathematical description for C we have a full description of the time evolution of f. The description of the effects of collisions on the distribution function was the major achievement of Ludwig Boltzmann (1844–1906). We will not give the form of the collision term here (it

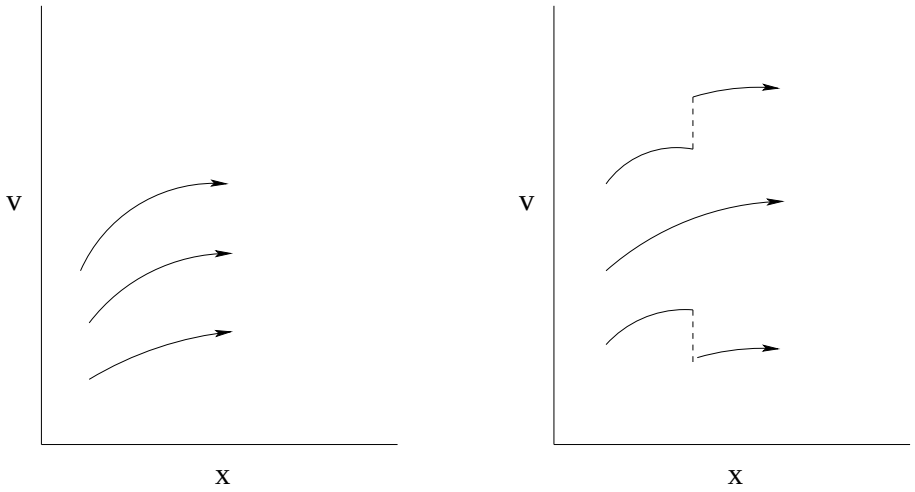

Figure 3: The effect of collisions on the distribution function. On the left: trajectories without collisions. On the right: a collision suddenly changes the trajectories of two particles.

can for example be found in the book by Choudhuri), but simply represent it with the symbol C. With the expression for C found by Boltzmann, Eq. 28 is known as the ***Boltzmann Equation***.

2.3.4 Stationary solution of the Boltzmann equation

It is useful to first consider the solution of the Boltzmann equation for the stationary case, without outside forces. For this case f does not change in time, so $\partial f/\partial t = 0$ and does not depend on position $f(\mathbf{x}, \mathbf{v}, t) = f(\mathbf{v})$, so $\partial f/\partial x = 0$. Without outside forces $\mathbf{a} = 0$.
All of this implies that the left hand side of Eq. 28 is zero, so the total collisional term C should also be zero. Boltzmann could show that for his expression for C this meant that f should be the Maxwell-Boltzmann distribution function, f_{MB}, given in Eq. 16.
The result is thus that the stationary solution for the Boltzmann equation is the Maxwell-Boltzmann distribution function. This is the reason why f_{MB} is so useful. Any initial condition left to itself will evolve to this distribution. Note however that its derivation relies on binary short range collisions, and the absence of non-conservative forces. When these conditions not hold, f_{MB} will not necessarily be the equilibrium solution.

Boltzmann could then also show that $f_{\rm MB}$ represents the state of maximum entropy, and thus that entropy always increases as a given initial state evolves towards the equilibrium solution $f_{\rm MB}$. This is known as Boltzmann's H-theorem, and is of course closely related to the 2nd law of thermodynamics, which says that in a closed system, the entropy can never go down. The fact that he was able to derive an irreversible relation from essentially reversible processes (binary collisions between particles), impressed many people and played an important part in convincing the scientific community of the correctness of Boltzmann's particle view (which was far from being universally accepted in the $19^{\rm th}$ century). There are however a number of subtleties connected to this derivation which have not stopped to generate discussions.

2.3.5 Macroscopic quantities

As mentioned above, we can derive macroscopic quantities such as density and energy from moments of the distribution function f (Eqs. 17–19). Let us consider a microscopic quantity Q and its mean value $\langle Q \rangle$ defined by

$$n\langle Q \rangle \equiv \int Q f \mathrm{d}\mathbf{v} \,. \tag{29}$$

The time evolution of Q can be found by multiplying both sides of the Boltzmann equation with Q and integrating over $\mathrm{d}\mathbf{v}$

$$\int Q \left(\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} \right) \mathrm{d}\mathbf{v} = \int Q C \mathrm{d}\mathbf{v} \,. \tag{30}$$

If χ is a quantity which is conserved in binary collisions, one can show that

$$\int \chi C \mathrm{d}\mathbf{v} = 0 \,. \tag{31}$$

This expresses the fact that since χ is conserved in collisions, the collisions cannot change χ. Using this and some manipulation of the LHS of Eq. 30, we can rewrite Eq. 30 as

$$\frac{\partial}{\partial t} n\langle \chi \rangle + \frac{\partial}{\partial x_i} n\langle v_i \chi \rangle - n \left\langle v_i \frac{\partial \chi}{\partial x_i} \right\rangle - n \left\langle a_i \frac{\partial \chi}{\partial v_i} \right\rangle - n \left\langle \frac{\partial a_i}{\partial v_i} \chi \right\rangle = 0 \,. \tag{32}$$

Here have used the suffix notation (see Sect. 1.2) to deal with the vectors $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{a}$. For an acceleration due to a conservative force, the last term ($\partial a_i / \partial v_i$) is also zero.[2]

[2]For a conservative force, the force and acceleration do not depend on the velocity of the particle.

We will now consider the three microscopic quantities conserved in binary conditions: mass, momentum and energy and from these find the equations for the evolution of macroscopic mass density, momentum density and energy density.

Mass

For $\chi = m$ the mass of the particles Eq. 32 gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0\,. \tag{33}$$

This is the ***continuity equation***, which we already found above from mass conservation principles (Eq. 14). Here we have defined the mass density $\rho = nm$ and the mean (or bulk) velocity of the particles as $\mathbf{u} = \langle \mathbf{v} \rangle$, the macroscopic gas velocity.

Momentum

For $\chi = mv_j$ the particle momentum in the j direction we get

$$\frac{\partial \rho \langle v_j \rangle}{\partial t} + \frac{\partial}{\partial x_i}(\rho \langle v_i v_j \rangle) - \rho a_j = 0\,. \tag{34}$$

Now $\langle v_j \rangle = u_j$, the macroscopic gas velocity, but $\langle v_i v_j \rangle \neq u_i u_j$. Let us define the difference between a particle velocity and the mean velocity as $w_i = v_i - u_i$. Obviously $\langle w_i \rangle = 0$. We then construct a tensor P as

$$P_{ij} = \rho \langle w_i w_j \rangle = \rho(\langle v_i v_j \rangle - u_i u_j)\,. \tag{35}$$

We can then write

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i u_j = -\frac{\partial P_{ij}}{\partial x_i} + \rho a_j\,. \tag{36}$$

This is known as the ***momentum or Euler equation***.
The tensor P contains the information about the microscopic random velocities of the particles. As we will see below, in the equivalent macroscopic quantity is the pressure p. Note that Clarke & Carswell define a stress tensor σ_{ij} (Sect. 2.2, Eq. 2.9) which is related to P by $\sigma_{ij} = \rho \langle v_i v_j \rangle = P_{ij} + \rho u_i u_j$.

Energy

For $\chi = \frac{1}{2}mv^2$ the kinetic energy of the particles, it makes sense to divide this into the energy connected with the mean velocity $\frac{1}{2}mu^2$ and the remainder which can be written as $m\mathbf{u} \cdot \mathbf{w} + \frac{1}{2}mw^2$. As $\langle \mathbf{u} \cdot \mathbf{w} \rangle = \mathbf{u} \cdot \langle \mathbf{w} \rangle = 0$ since $\langle \mathbf{w} \rangle = 0$, the equation becomes

$$\frac{\partial}{\partial t}\frac{1}{2}\rho(u^2 + \langle w^2 \rangle) + \frac{\partial}{\partial x_i}\frac{1}{2}\rho \langle (u_i + w_i)|\mathbf{u} + \mathbf{w}|^2 \rangle = \rho \mathbf{u} \cdot \mathbf{a}\,. \tag{37}$$

The $\langle (u_i + w_i)|\mathbf{u} + \mathbf{w}|^2 \rangle$ term can be written out as follows

$$\langle (u_i + w_i)(u_i + w_i)^2 \rangle = u^2 u_i + 2\mathbf{u} \cdot \langle \mathbf{w} w_i \rangle + u_i \langle w^2 \rangle + \langle w_i w^2 \rangle \,. \tag{38}$$

We define the internal energy density of the gas as $\rho\mathcal{E} = \frac{1}{2}\rho\langle w^2 \rangle$ (the kinetic energy contained in the random motions of the particles; this is equal to half the trace of the tensor P) and define a vector $\mathbf{q}$ to be $\rho\langle \frac{1}{2} w^2 \mathbf{w} \rangle$ (this will turn out to be the heat flux due to conduction). We can then write

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial x_i} u_i \left(\frac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i} u_j P_{ij} = \rho \mathbf{u} \cdot \mathbf{a} \,, \tag{39}$$

the ***energy equation***. We will call the sum of the kinetic energy density and the internal energy density $\frac{1}{2}\rho u^2 + \rho\mathcal{E}$ the total energy density E.

2.3.6 Closure relation

We have thus obtained three general equations describing the evolution of mass, momentum and energy in a gas:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{40}$$

$$\frac{\partial \rho u_i}{\partial t} + \boldsymbol{\nabla} \cdot (\rho u_i \mathbf{u}) = -(\boldsymbol{\nabla} \cdot \mathsf{P})_i + \rho a_i \tag{41}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E + \mathsf{P})\mathbf{u} + \boldsymbol{\nabla} \cdot \boldsymbol{q} = \rho \mathbf{u} \cdot \mathbf{a} \tag{42}$$

These three equations are general but not a closed set since we have 5 equations, but 13 quantities[3]: ρ, $\mathbf{u}$, $\mathcal{E}$, and P.
What can we do with the excess unknowns? In order to get a set of solvable equations we need to get rid of them. They seem to be related to the microscopic behaviour of the fluid, and so we need the distribution function to say something about them. For a gas which is at equilibrium, this is f_{MB}, but such a gas would not evolve any more. Instead, let us assume a situation in which the gas *locally* has a Maxwell-Boltzmann distribution. This means that the gas consists of regions ('fluid elements' as we called them in Sect. 1.1) which each have a Maxwell-Boltzmann distribution, but which can differ in n, T and $\mathbf{u}$. For this assumption

[3]Note that $\rho\mathcal{E} = \frac{1}{2}\rho\langle w^2 \rangle$ and that $P_{ij} = \rho\langle w_i w_j \rangle$, so that $\sum_i P_{ii} = 2\rho\mathcal{E}$, thus we have 13 quantities, not 14.

to hold there need to be enough collisions within such a region. As we saw above, this means that the mean free path $\lambda_{\rm mfp}$ should be smaller than the size of the region L: $\lambda_{\rm mfp} \ll \ell$.
We can now evaluate the tensor P using the Maxwell-Boltzmann distribution function and thus reduce the number of unknowns. Since $P_{ij} = \rho\langle w_i w_j\rangle$ and $\langle Q\rangle = \int Q f \mathrm{d}\mathbf{v}/n$, we find that locally

$$P_{ij} = \rho\left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2}\int \mathrm{d}\mathbf{w} w_i w_j \exp\left(-\frac{mw^2}{2k_{\rm B}T}\right)\,. \tag{43}$$

Since we integrate symmetrically over *all* velocities, both positive and negative, the off-diagonal terms, being asymmetric around zero, give no contribution under the integral over $\mathbf{w}$, and we are left with only the diagonal terms, for which the integral gives

$$P_{ij} = nk_{\rm B}T\delta_{ij}\,, \tag{44}$$

a diagonal matrix. As we know from thermodynamics, the term $nk_{\rm B}T$ is the gas pressure p. So $P_{ij} = p\delta_{ij}$. Since we defined $\rho\mathcal{E}$ to be $\frac{1}{2}\rho\langle w^2\rangle$, we also find that $\rho\mathcal{E} = \frac{3}{2}nk_{\rm B}T$, showing that only the temperature determines the specific internal energy density $\mathcal{E}$ of the gas, as $\rho = nm$. We have thus recovered the Equation of State for an ideal gas by averaging over the Maxwell-Boltzmann distribution.
Similar considerations show that for the Maxwell-Boltzmann distribution function the heat conduction flux $\mathbf{q}$ found in the energy equation, is equal to zero (since it involves a symmetric integral over the asymmetric function $w_i w^2$).
We have thus removed the excess unknowns and obtained a closed set of equations, known as the set of **Euler equations for an inviscid fluid**. The word inviscid means without viscosity:

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\mathbf{u} = 0\\
&\frac{\partial\rho u_i}{\partial t} + \boldsymbol{\nabla}\cdot\rho u_i\mathbf{u} = -\nabla_i p + \rho a_i\\
&\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(E+p)\mathbf{u} = \rho\mathbf{u}\cdot\mathbf{a}\,.
\end{aligned} \tag{45}$$

These equations are not the full fluid equations, since they assume that $f = f_{\rm MB}$ everywhere (although not necessarily the *same* $f_{\rm MB}$ everywhere). Obviously in

real fluids there will be deviations from f_{MB}, and these give rise to so-called 'transport phenomena' such as viscosity and thermal conduction. However, in astrophysics these transport phenomena are often unimportant and many astrophysical systems can be described with the set of Euler equations. We will later return to consider the terms introduced by deviations from a local Maxwell-Boltzmann distribution (Chapter 8).
It is important to realize that we have removed the information about the motions of the individual particles through the introduction of the tensor P, which in the Euler equations is represented by the pressure terms. So, you could say that the appearance of pressure terms is our punishment for not wanting to deal with the motions of individual particles.

2.4 Summary

The following points summarize the most essential concepts introduced in this chapter.

- When considering the state of gas from the microscopic point of view (as a collection of particles), we introduce the so-called distribution function which specifies how many particles have a certain position and velocity. The 6-dimensional space of position and velocity is known as *phase space*. The distrubution function is thus a density function in phase space.

- The time evolution of the distribution function for a dilute gas is given by the Boltzmann equation (Eq. 28).

- The Boltzmann equation has a collision term C which contains the effects of collisions between particles. The equilibrium solution of the Boltzmann equation is the Maxwell-Boltzmann distribution function (Eq. 16), which is also the state of maximum entropy.

- Position-dependent macroscopic gas quantities can be recovered by integrating the distribution function over velocity space. The time evolution of these macroscopic quantities can be found by taking moments of the Boltzmann equation. For the collisionally invariant quantities (mass, momentum and kinetic energy) these moments result in the equation of gas dynamics.

- To obtain a closed set of equations from the moment equations, one needs to define a closure relation.

- The equation of gas dynamics simplify into the Euler equations for inviscid flow if one assumes for a closure relation that the gas is everywhere in *local* equilibrium (i.e. it consists of fluid elements which each have a Maxwell-Boltzmann distribution).

3 Equations is Power?

We have now obtained a set of equations which describe the time evolution of a gas (Eqs. 45). Before proceeding it is good to consider this set of equations.
It is a set of *partial differential equations* in $(\mathbf{x}, t)$. The equations in the set are coupled in the sense that for example to solve for the time evolution of ρ you need to know $\mathbf{u}$ which has its own equation for its time evolution. In addition the equations are non-linear due to the double appearance of $\mathbf{u}$ in the $\boldsymbol{\nabla} \cdot \rho u_i \mathbf{u}$ term in the momentum equations. These properties make the equations hard to solve in the most general cases, but at the same time make for interesting solutions. Consider a complex flow pattern which you may have seen in a river: this is a solution of these equations (or at least some equations very similar to them).
Since it is hard to find solutions in general cases, most analytical solutions for the gas dynamic equations are for very simplified cases. We will encounter a number of these solutions later. As you will notice, the simplifications often take the shape of a reduction of the dimensionality of the problem (using assumed symmetries in the flow) or a reduction in the number of quantities to solve for (by assuming simple relations between different quantities, for example between pressure and density).
Because of the very limited set of possible analytical solutions, the development of techniques to find numerical solutions to the fluid equations has revolutionized the field. Computational gas dynamic simulations allows one to consider much more complex cases, such as star formation, galaxy formation, etc. One has however constantly to be aware of the limitations of the numerical solutions which will always have a finite resolution. Chapter 7 will introduce some of the concepts of computational fluid dynamics.

3.1 Advection

It can be seen that all five equations of the set of Euler equations for inviscid flow have some terms in common. Each of them can be written as

$$\frac{\partial Q}{\partial t} + \boldsymbol{\nabla} \cdot Q\mathbf{u} = \text{other terms} \tag{46}$$

The first term is the change of Q at position $\mathbf{x}$ and the second term has to do with the flow velocity $\mathbf{u}$. If the "other terms" are zero one can show mathematically that this equation represents a quantity Q being carried along by the flow $\mathbf{u}$. This

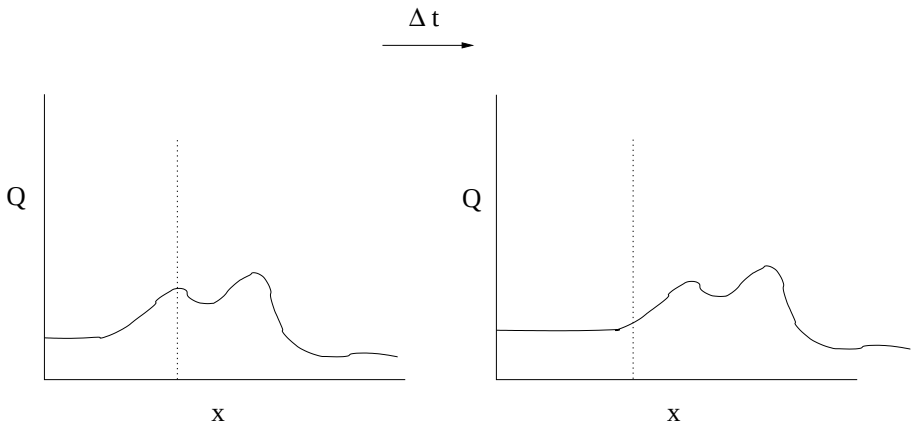

Figure 4: The concept of advection. After a time Δt the pattern in quantity Q has been carried to the right by the flow.

process is known as ***advection*** and therefore the term $\boldsymbol{\nabla} \cdot Q\mathbf{u}$ is known as the advection term.
Figure 4 illustrates how a pattern in the quantity Q is carried along by a flow. In this case the velocity u is the same at all positions and the pattern is simply shifted to the right. If u would be different at different positions, the pattern would be shifted *and* deformed, either compressed or stretched, depending on the shape of $u(x)$.
One could thus write the Euler equations thus

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u_i\\ E\end{pmatrix}}_{\text{change in time}} + \underbrace{\boldsymbol{\nabla}\cdot\begin{pmatrix}\rho\mathbf{u}\\ \rho u_i\mathbf{u}\\ E\mathbf{u}\end{pmatrix}}_{\text{advection}} = \underbrace{\begin{pmatrix}0\\ -\nabla_i p + \rho a_i\\ -\boldsymbol{\nabla}\cdot(p\mathbf{u}) + \rho\mathbf{a}\cdot\mathbf{u}\end{pmatrix}}_{\text{other processes}} \tag{47}$$

The other processes here are the pressure forces and the acceleration due to external forces.

3.2 Momentum or velocity?

The equations of fluid dynamics are coupled, and therefore it is also possible to write them in many different forms. As an example one can consider the momentum equation which describes the time evolution of the momentum density $\rho\mathbf{u}$. One might prefer to write this as an equation for the time evolution of the velocity

$\mathbf{u}$ instead. This is possible by using the chain rule and the continuity equation. Since

$$\frac{\partial \rho u_i}{\partial t} = \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t}\,, \tag{48}$$

and

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \mathbf{u})\,, \tag{49}$$

one finds that

$$\rho \frac{\partial u_i}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla}) u_i = -\nabla_i p + \rho a_i\,. \tag{50}$$

The latter equation is found in Clarke & Carswell as Eq. (2.17) on page 17. However, there it is written for $\mathbf{u}$ instead of for velocity component u_i. Because of this the second term becomes ambigious: it is unclear how the term $\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}$ should be interpreted. Equation 50 shows how it should be interpreted.
Up to now we have written the momentum equation as an equation for the velocity component u_i. If one wants to write the momentum equation as an equation for the velocity vector $\mathbf{u}$, one should introduce the dyadic tensor $\mathbf{u} \otimes \mathbf{u}$ (see Sect. 1.2). With it one can then write

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\boldsymbol{\nabla} p + \rho \mathbf{a}\,, \tag{51}$$

where the divergence operator $\boldsymbol{\nabla}\cdot$ now works on a two-dimensional tensor, reducing it to a vector.

3.3 Eulerian versus Lagrangian

When following the time evolution of gas quantities such as the density, pressure and velocity, there are two different points of view one can take. These are

1. **Eulerian description:** one defines a fixed coordinate grid in space and follows how the gas quantities are changing at a given position. This is the perspective of someone sitting at one position and making measurements. If a steadily moving density wave is passing, a time evolution of the density is found. When considering the time derivative in this description, the symbol ∂ is used. This is the point of view which we have used up to now.

2. **Lagrangian description:** one chooses a fluid element in the fluid and follows how its properties change. This is the perspective of someone moving with the fluid and making measurements. In this case a steadily moving

density wave would not be described as having a time evolution as we now have the perspective of someone "riding the wave". When considering the time derivative in this description, the symbol D is used.

The two point of views are related through the following expression (for a general quantity Q):

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} Q \,. \tag{52}$$

This expression can be derived by following a fluid element over infinitesimally small time step δt. If it starts at position $\mathbf{r}$ at time t, then at time $t + \delta t$ it will be at position $\mathbf{r} + \delta\mathbf{r}$. The Lagrangian derivative is thus

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \left[\frac{Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right] . \tag{53}$$

The numerator can be written as

$$\begin{aligned} &Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t) = \\ &\quad Q(\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t) + Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t + \delta t) \,, \end{aligned} \tag{54}$$

which to first order in δt and $\delta\mathbf{r}$ equals

$$= \frac{\partial Q(\mathbf{r}, t)}{\partial t} \delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q(\mathbf{r}, t + \delta t) \,. \tag{55}$$

The second term can then be expanded as

$$\delta\mathbf{r} \cdot \left[\boldsymbol{\nabla} Q(\mathbf{r}, t) + \frac{\partial \boldsymbol{\nabla} Q}{\partial t} \delta t \ldots \right] \tag{56}$$

For δt and $\delta\mathbf{r} \to 0$ the time derivative of the gradient of Q disappears, as it is a second order term. Realizing that the fluid velocity $\mathbf{u} = \delta\mathbf{r}/\delta t$, we thus obtain Eq. 52.

It is instructive to transform the fluid equations to their Lagrangian form, since this provides us with the perspective of fluid elements. Applying Eq. 52 to the continuity equation, the Lagrangian form becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\boldsymbol{\nabla} \cdot (\rho\mathbf{u}) + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{u} \,, \tag{57}$$

or

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{58}$$

This equation shows that the density of a fluid element only changes when $\boldsymbol{\nabla} \cdot \mathbf{u} \neq 0$. So-called incompressible flows have the property $\mathrm{D}\rho/\mathrm{D}t = 0$ and as a consequence their velocity field has to obey $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, i.e. it must be divergence free. It is a useful exercise to try to reason why this must be the case; consider what velocity differences mean for the volume of a fluid element...
The Lagrangian form can be seen to remove the advection term since one is moving with the fluid. This sometimes means that the Lagrangian form is the simplest one. It is for example possible to rewrite the momentum equation in Lagrangian form as

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho \mathbf{a} \,, \tag{59}$$

from which the connection with Newton's 2$^{\mathrm{nd}}$ law becomes apparent

$$m\mathrm{a} = \mathrm{F} \,. \tag{60}$$

It is therefore one sometimes speaks of pressure forces working on the fluid. Note however that it would be more appropriate to speak of pressure *gradient* forces.

3.4 Particle paths

Even though the Lagrangian form may be sometimes easier to deal with, one should realize that for a full solution, one still needs to solve for the paths the fluid elements are taking. If one wants to trace the trajectories of fluid elements one needs to solve the equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathrm{u}(\mathbf{r}, t) \,, \tag{61}$$

which describes the path of a fluid element, or a so-called particle path.

3.5 Gravity

Up to now we have carried along an external acceleration denoted by $\mathbf{a}$. In practice this external acceleration is almost always a gravitational one. Therefore we replace the symbol $\mathbf{a}$ by $\mathbf{g}$.
Since gravity is a conservative force, it can be described by a scalar potential field Ψ such that

$$\mathbf{g} = -\boldsymbol{\nabla} \Psi \tag{62}$$

Since it is only the gradient that matters, an arbitrary scalar can always be added to the field. Usually the field is normalized this way so that $\Psi(\infty) = 0$.

The gravitational field may be completely external, for example the field imposed by a planet or a star on its surroundings, or it may be internal. The latter case is usually called *self-gravity*: a gas cloud feels the gravitational field produced by its own mass. To find the gravitational field of a mass distribution ρ one needs to solve the Poisson equation:

$$\nabla^2 \Psi = 4\pi G \rho \,, \tag{63}$$

where G is the gravitational constant. The Poisson equation is completely equivalent to the equation of the electric potential of a charge density field, familiar from electromagnetism. A derivation can be found in Clarke & Carswell (Sect. 3.2).
Solving the Poisson equation is rather complicated as it is a second order differential equation. Even quite simple density distributions can produce rather complicated potentials. If you are interested to see some examples, you can read Chapter 3 of Clarke & Carswell.
A useful theorem related to the self-gravitating systems is the *Virial Theorem*. Since gravity is only an attractive force, a collection of self-gravitating particles needs to have some internal motions in order not to collapse. However, if the velocities of the particles are too large, the collection will disperse. The Virial Theorem specifies the amount of kinetic energy that is needed for a collection of particles not to collapse and not to fly apart. If T is the total kinetic energy of the particles and Ω their collective gravitational potential energy, then the Virial Theorem states that

$$2T + \Omega = 0 \,. \tag{64}$$

Note that T is the total kinetic energy, so for a gas it consists of the kinetic energy of the mean motion, and the kinetic energy contained in random motions, that is, the internal energy of the gas.

Read Sections 3.5 and 3.6 of Clarke & Carswell to learn more about the Virial Theorem.

The Virial Theorem is an important concept in astrophysics. It relates the gravitational potential energy and the kinetic energy of any self-gravitating systems in equilibrium. This includes gas clouds, but also stellar systems and galaxy systems. Since the gravitational potential energy of the system is proportional to its mass squared, it becomes possible to 'weigh' systems by measuring their kinetic energy content. This was for example used in the 1930s to find the mass of galaxy clusters. These measurements provided the first indication that a large part of the mass of galaxy clusters is 'dark'. In current cosmology this is interpreted as being due to the presence of dark matter in the Universe.

3.6 Summary

These are the most important points addressed in this chapter

- The set of fluid equations is a set of couples, partial and non-linear differential equations. Analytical solutions are only available for simpler problems.
- The term $\boldsymbol{\nabla} \cdot Q\mathbf{u}$ found in all five equations describes the advection of quantity Q.
- The Eulerian description of a fluid gives the evolution of the fluid quantities at fixed points $\mathbf{r}$ in space. It is associated with the differential symbol ∂. The Lagrangian description gives the evolution of the fluid quantities in fluid elements which are carried along by the flow. It is associated with the differential symbol D.
- The gravitational acceleration $\mathbf{g}$ can be calculates from the gravitational potential Ψ. If the gas is self-gravitating, Ψ must be calculated from the Poisson equation.
- For a self-gravitating system to be in equilibrium the Virial Theorem hold which states that twice the kinetic energy of the system should equal -Ω, the gravitational potential energy of the system.

4 Finding order in chaos

In Chapter 2 we derived the energy equation from the microscopic view, showing that for the inviscid flow it is given by

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E + p)\mathbf{u} = \rho \mathbf{u} \cdot \mathbf{g}\,, \tag{65}$$

with $E = \frac{1}{2}\rho u^2 + \rho\mathcal{E}$. The first term in E is the kinetic energy density of the flow, corresponding to the mean motion of the particles, the second term is the internal energy density of the flow, corresponding to the random motions of the particles. Since only the total energy of the particles is conserved, for the most general case one has to consider the evolution of the sum of these two energy densities. As we will see later, flows can convert one type of energy into the other.
The internal energy of a gas is sometimes also called the thermal energy. It is the internal energy which is a key concept in thermodynamics, where most of the time the motion of the gas is not considered. In this chapter we deal with several concepts connected with the internal energy of a gas, concepts which may already be known to you from studying thermodynamics.

4.1 Energy equation and relation to thermodynamics

When in Chapter 2 we assumed that the distribution function of the particles is locally a Maxwell-Boltzmann distribution function, this also implied that the gas is ideal. For an ideal gas the specific internal energy $\mathcal{E}$ is only a function of temperature, and *not* of the density. This relation between $\mathcal{E}$ and other thermodynamic quantities is called the *equation of state* (EOS). As we saw in Chapter 2, the specific internal energy and the pressure for a monatomic gas are related through $\mathcal{E} = \frac{3}{2}p/\rho$, so instead of the internal energy, the EOS is usually given as a relation between the pressure and the other thermodynamic quantities of the gas. In fact, the assumption of a Maxwell-Boltzmann state of the gas already gave us an EOS: $p = nk_{\mathrm{B}}T$. This is the EOS of an ideal gas. It shows that for an ideal gas the internal energy $\mathcal{E}$ is only a function of the temperature: $\mathcal{E}_{\mathrm{ideal}} = \mathcal{E}_{\mathrm{ideal}}(T)$.
If $\mathcal{E}$ starts to depend on other quantities, such as the density, the gas is no longer considered to be ideal. From the course on stellar structure you may remember that the cores of stars can consist of a degenerate gas. If the gas is fully degenerate, the internal energy depends *only* on the density ($p \propto \rho^{\frac{5}{3}}$) and no longer on the temperature, so this is a very non-ideal gas.

Read the beginning of Chapter 4 up to Section 4.2 of Clarke & Carswell to learn about the the equation of state.

As given in Sect. 4.1 of Clarke & Carswell, for ideal gas the equation of state can be written as

$$p = \frac{\mathcal{R}_*}{\mu_u}\rho T \tag{66}$$

where $\mathcal{R}_* = 8300$ J K^{-1} kmole^{-1} (and equals 1000$\mathcal{R}$, the gas constant) and μ_u is the mean molecular weight in atomic mass number (1/12 of the mass of ^{12}C, or 1.661×10^{-27} kg). The Boltzmann constant is connected to the gas constant through $k_{\rm B} = \mathcal{R}/N_{\rm A}$, where $N_{\rm A}$ is Avogadro's number (6.022×10^{23} mole^{-1}). This form of the equation of state is the usual form for Earth-based applications. However, in astrophysics, hydrogen is by far the dominant element and it is more usual to use the mass of hydrogen as the unit mass ($m_{\rm H} = 1.672 \times 10^{-27}$ kg). In this case the the equation of state is usually written as

$$p = \frac{k_{\rm B}}{\mu m_{\rm H}}\rho T\,, \tag{67}$$

where μ is the molecular weight expressed in $m_{\rm H}$. Since ρ is the mass density of the gas, this is equivalent to $p = nk_{\rm B}T$ since the mass and number density of gas particles are connected through

$$\rho = \mu m_{\rm H} n\,. \tag{68}$$

4.2 Barotropic equations of state

The energy equation Eq. 65 is the most general equation for the evolution of the total energy density of a gas (consisting of internal + kinetic energy density). However, in many cases it is not actually necessary to solve the energy equation because the pressure is only a function of density, $p = p(\rho)$. Since the pressure gives $\mathcal{E}$, both can then be derived from the solution of the density. The kinetic energy can always be found from the momentum equation. If $p = p(\rho)$, the EOS is called *barotropic*. There are two main cases of such a barotropic EOS

1. **isothermal EOS:** the temperature is (locally) constant, and $p \propto \rho$. This can happen if other processes than the flow dominate the thermal state of the gas. These other processes (heating and cooling, for example through radiative processes) can act as a *thermostat*, keeping the temperature of the gas constant despite changes in the gas.

2. **adiabatic EOS:** the gas only undergoes reversible changes and the entropy of fluid elements is conserved. In this case $p \propto \rho^\gamma$ where γ is the adiabatic index.

Read the Section 4.2 of Clarke & Carswell to learn about these two cases of barotropic EOS.

The adiabatic index for a monatomic gas is $\gamma = \frac{5}{3}$, and for a gas made of diatomic molecules it is $\gamma = \frac{7}{5}$.
As we wrote above, for an ideal gas the specific internal energy $\mathcal{E}$ only depends on the temperature. How can it then be that for that same ideal gas we now find that the pressure (proportional to $\rho\mathcal{E}$) depends on ρ^γ? The answer is that $\mathcal{E}$ still only depends on the temperature, but that the temperature has to follow the density ($T \propto \rho^{\gamma-1}$) when the entropy is conserved. So the dependency of $\mathcal{E}$ on the density is an *indirect* one, not a direct one (as it is for the case of a fully degenerate gas that was mentioned above).

4.3 Deriving the energy equation from thermodynamics

When one starts with the macroscopic theory of thermodynamics, one can also derive the energy equation. This is done by considering the 1st law of thermodynamics, which states that energy is conserved. Thermodynamics only considers the internal energy of the gas ($\mathcal{E}$), whereas the particle perspective considers the total energy of the gas, the sum of kinetic and internal energy. However, the evolution of the kinetic energy can be found from the momentum equation and then be added to the evolution of the internal energy.
The first law of thermodynamics can be written as

$$\mathrm{d}\mathcal{E} + p\mathrm{d}V = \mathrm{d}Q\,, \tag{69}$$

stating that the internal energy $\mathcal{E}$ can only change due to heat being added or removed ($\mathrm{d}Q$) or through work done by the gas when it changes its volume ($p\mathrm{d}V$ term). All these term actually come back in the enery equation. The $p\mathrm{d}V$ term is contained in the $\boldsymbol{\nabla}\cdot(p\mathbf{u})$ term which can be split into $p\boldsymbol{\nabla}\cdot\mathbf{u}$ (internal energy change due to compression/expansion; as we saw above in the Lagrangian version of the continuity equation, $\boldsymbol{\nabla}\cdot\mathbf{u}$ corresponds to volume changes in the flow) and $\mathbf{u}\cdot\boldsymbol{\nabla}p$ (kinetic energy change due to pressure gradients).

The Q term is new for us, since it represents external processes which add or remove energy from the gas. Below we will consider this term in a little bit more detail.

> Read the Section 4.3 of Clarke & Carswell to learn about how the energy equation derives from the 1st law of thermodynamics.

There is one small difference between the book and the derivation from Chapter 2. The book defines the total energy density as $\frac{1}{2}\rho u^2 + \rho\mathcal{E} + \rho\Psi$, where the last term is the potential energy term. If we call this definition of the total energy density E^*, the energy equation becomes

$$\frac{\partial E^*}{\partial t} + \boldsymbol{\nabla} \cdot (E^* + p)\mathbf{u} = \rho \frac{\partial \Psi}{\partial t} \, ,$$

as shown in Eq. (4.32) of Clarke & Carswell. This formulation is completely equivalent to Eq. 65 as can be seen from some manipulation:

- The term $\rho\partial\Psi/\partial t$ occurs on both sides of equation and thus cancels.
- In the advection term for E^* there is a $\boldsymbol{\nabla} \cdot (\Psi\mathbf{u})$ which can be rewritten as $\rho\mathbf{g} \cdot \mathbf{u}$, since $\mathbf{g} = -\boldsymbol{\nabla}\Psi$. This is the term we had in the energy equation for E.
- The remaining terms $\Psi\partial\rho/\partial t$ and $\Psi\boldsymbol{\nabla} \cdot (\rho\mathbf{u})$ cancel each other due to the continuity equation.

So we can conclude that the two versions of the energy equation are equivalent. This is another example of how the fluid equations can be rewritten in different forms. A further example is writing down separate equations for the evolution of the internal and kinetic energy density, a problem which we consider in one of the exercises.

4.4 Heating and cooling processes

The energy equation Eq. 65 describes the situation when only fluid processes change the energy density. However in astrophysics the energy density is often modified by other processes, such as radiative processes. To accommodate this in

the equation, Clarke & Carswell introduce a term $-\rho\dot{Q}_{\rm cool}$ which when added to our energy equation gives

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E + p)\mathbf{u} = \rho\mathbf{u} \cdot \mathbf{g} - \rho\dot{Q}_{\rm cool} \, .$$

If $\dot{Q}_{\rm cool} > 0$ this represents energy loss and if $\dot{Q}_{\rm cool} < 0$ it represents energy gain. It is important to realize that $\dot{Q}_{\rm cool}$ only affects $\mathcal{E}$, the (specific) internal energy, and obviously not the kinetic energy of the gas.
Clarke & Carswell describe four processes that can contribute to energy *transport*

1. Cosmic Rays
2. Conduction
3. Convection
4. Radiation

Of these only cosmic rays and radiation are actual heating and cooling processes that contribute to $\dot{Q}_{\rm cool}$. As we will see in Chapter 8, *conduction* is a fluid process that is caused by deviations from the assumption of a local Maxwell-Boltzmann distribution. It is therefore an *internal* fluid process, not an external source or sink of energy.
Convection is a complex macroscopic fluid process, which is in principle already described by the fluid equations. Because of its complexity it is sometimes added to the equations in a parametrized form, which is why Clarke & Carswell group it under energy transport processes.

Read the Section 4.4 of Clarke & Carswell to learn about energy transport processes. The details of the various radiation processes described on pages 42 – 45 are optional.

The interaction between gas and radiation is more fully treated in the course on the Interstellar Medium, and therefore not part of this course. The main message to remember here is that there can be a $\dot{Q}_{\rm cool}$ term in the energy equation representing energy gains and losses due to interactions with cosmic rays and radiation. Often this interaction works as a thermostat, fixing the temperature of the gas, allowing the use of the isothermal EOS (see Sect. 4.2). It is for example often a good assumption that a gas being photo-ionized and heated by a hot star is isothermal

at a temperature of $\sim 10^4$ K, as the photons can easily heat the gas to temperatures above 10^4 K, but radiative cooling processes will reduce the temperature to values below which radiative cooling becomes inefficient, which happens to be $\sim 10^4$ K. Radiative cooling is a very importent process in astrophysics. The reason is that under many circumstances the density of the gas is so low that if due to collisions, a molecule, atom or ion gets excited, it is more likely to de-excite by sending out a photon then through another collision. If this photon does not get absorbed, it will leave the gas, carrying with it some energy. In fully ionized gases, the main cooling process is the electrodynamic interaction between charged particles, mostly between electrons and ions. This is known as free-free emission or bremsstrahlung (this concept will come back in the computer lab).
Radiative heating is mostly through ionization processes, either through photons or collisions with cosmic rays. The excess energy beyond the ionization energy needed to release the electron, is given to the electron who then through collisions can share this with the rest of the particles in the gas. Another path for radiative heating is through dust grains. They can absorb photons and increase their temperature. Interaction with gas phase particles then leads to a heating of the entire mixture of gas and dust.

4.5 Adiabatic flow and entropy

When the gas has an adiabatic EOS, $p \propto \rho^\gamma$, the entropy of fluid elements is conserved. This implies that the entropy of a fluid element is a function of $p\rho^{-\gamma}$. A more thorough analysis shows that the thermodynamic quantity of entropy is given by

$$s = C_{\rm V} \ln\left(\frac{p}{\rho^\gamma}\right) + s_0 \tag{70}$$

where $C_{\rm V}$ is the specific heat capacity at constant volume and s_0 is a normalization constant. The adiabatic index is given by $\gamma = C_{\rm p}/C_{\rm V}$, where $C_{\rm p}$ is the specific heat capacity at constant pressure.
Saying that the entropy is conserved is equivalent to saying that no irreversible processes are affecting the gas. This follows from rewriting the first law of thermodynamics in terms of the entropy.
Irreversible processes can be external processes that add energy to or remove energy from the gas, processes contained in the $\dot{Q}_{\rm cool}$ term. However, even when no external heating and cooling processes are operating, irreversible changes can happen in a fluid due to shock waves. We will treat shocks in more detail in Chap-

ter 6. We will see that in shocks kinetic energy is converted into thermal energy, which is an irreversible process that increases the entropy. So, across a shock entropy is not conserved.
Further irreversible processes are those connected with transport processes, to be treated in Chapter 10. These processes are associated with viscosity and heat conduction. If these are important, the gas will also not conserve entropy.
The summary is thus that if a gas is inviscid (no viscosity and conduction) and external heating and cooling processes are unimportant, then away from shocks, the gas will have an adiabatic EOS (as explained above in Sect. 4.2). In this case it is not necessary to solve the energy equation, since the pressure (internal energy) can be derived from the density and the kinetic energy from the momentum density.

4.6 Pressure, internal energy and enthalpy

The adiabatic relation also allows us to derive the connection between pressure p and internal energy $\mathcal{E}$. If the flow is adiabatic, the first law of thermodynamics can be written as

$$\mathrm{d}\mathcal{E} + p\mathrm{d}V = 0\,, \tag{71}$$

and we also know that for this condition

$$p = K\rho^{\gamma}\,, \tag{72}$$

with K some constant. Combining them gives

$$\begin{aligned} \mathrm{d}\mathcal{E} &= -p\mathrm{d}V = K\rho^{\gamma}\mathrm{d}\left(\rho^{-1}\right) = \frac{K}{\gamma-1}\mathrm{d}\rho^{\gamma-1} = \frac{1}{\gamma-1}dp/\rho & (73)\\ &\Rightarrow & \\ &\mathrm{d}\mathcal{E} = d\frac{p/\rho}{\gamma-1} & (74)\\ &\Rightarrow & \\ &\mathcal{E} = \frac{p/\rho}{\gamma-1} & (75) \end{aligned}$$

Although derived from the assumption of adiabatic conditions, the internal energy $\mathcal{E}$ of a fluid is an intrinsice property of the fluid, not depending on the actual conditions. So this relation for $\mathcal{E}$ is generally valid. In fact, for a monatomic gas, $\gamma = 5/3$ and thus $\mathcal{E} = \frac{3}{2}p/\rho$, something we already found in Chapter 2.

Another energy related quantity which is sometimes used is the *enthalpy*. The specific enthalpy is defined as

$$h = \mathcal{E} + p/\rho\,. \tag{76}$$

Using Eq. 75 this can also be written as

$$h = \frac{\gamma}{\gamma - 1} p/\rho\,. \tag{77}$$

4.7 Summary

The most important points to remember from this chapter are

- The general form of the EOS for an ideal gas is Eq. 67.
- The adiabatic index γ (ratio of the specific heat capacities C_p/C_V), connects the specific internal energy density $\mathcal{E}$, pressure and density through the relation $\rho\mathcal{E} = p/(\gamma - 1)$. It also occures in the definition of the entropy which is proportional to $\ln(p\rho^{-\gamma})$.
- Under some circumstances, the pressure can be written as a function of the density (barotropic EOS), removing the need to solve for the energy equation.
- The two well-known cases of barotropic flow are isothermal ($p \propto \rho$) and adiabatic ($p \propto \rho^{\gamma}$).
- External processes (mostly radiative) can add or remove internal energy from the gas.

5 Spherical Cows

There is a well-known joke which says that if a theoretician would be asked to study how a cow produces milk, they would start by assuming a spherical cow (see Fig. 5). In this chapter we will look at some simple, static solutions of the gas dynamic equations, some of which assume spherical symmetry. In the rest of the course we will encouter other examples of "spherical cows" since reducing the dimensionality of the equations is one way to simplify them so that finding analytical solutions becomes possible.

Figure 5: A typical spherical cow model.

5.1 Non-cartesian coordinates

We have been writing the Euler equations for inviscid flow in general terms without specifying the coordinate system to be used, employing the nabla symbol ∇ to represent the differential operators. The equations are of course valid in any coordinate system, and thus writing them in general terms is appropriate. However, when one starts to look for solutions for specific cases (supernova explosion, an accretion disk, a stellar wind), one has to choose a certain coordinate system. The mathematically simplest one is of course the cartesian coordinate system, usually denoted with x, y, z. In this type of coordinate system the various vector and tensor operators have their simplest form.

However, in many cases the symmetry of the problem one is trying to solve suggests other, so-called curvilinear coordinate systems. The two most common ones are cylindrical coordinates (more fully described as cylindrical polar coordinates), often represented by the symbols (R, ϕ, z) and spherical coordinates (spherical polar coordinates), often represented by the symbols (r, θ, ϕ). These systems are called curvilinear coordinate systems because at least one coordinate follows a *curved* path in a cartesian representation.
For example, an idealized supernova explosion proceeds spherically, and the wind from a spherical star will also travel radially outward. In these cases it is more appropriate and even simpler to use a spherical coordinate system. Similarly, it is easier to use cylindrical coordinates to describe a cylindically symmetric accretion disk.
Because of the inherent curvature in curvilinear coordinates, the vector and tensor operators obtain different forms. The hand-out *DIFFERENTIAL OPERATORS IN CURVILINEAR COORDINATES* gives an overview of these forms for the cylindrical and spherical cases. One thing to keep in mind when looking at these forms is that the angles do not have dimension of length, so for dimensional reasons differentials with respect to angle have to be divided by some quantity of dimension length. Also note that some of the operators contain terms which are *not* differentials.
We will consider the case of cylindrical polar coordinates in a later chapter when we will look at accretion disks, which we will describe as two-dimensional objects in (R, ϕ). The purpose of using curvilinear coordinates is to make use of possible symmetries in a system, and this often means that one does not use the full three-dimensional set; a symmetry allows one to reduce the dimensionality of the system. One often occuring symmetry is *spherical* symmetry. In this case the only coordinate is the spherical radius r. The Euler equations for spherical symmetry become

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r\right) = 0 \\
&\frac{\partial \rho u_r}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r^2\right) = -\frac{\partial p}{\partial r} + \rho\frac{\partial \Psi}{\partial r} \\
&\frac{\partial E}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 (E+p) u_r\right) = \rho u_r \frac{\partial \Psi}{\partial r}\,.
\end{aligned} \tag{78}$$

5.2 Hydrostatic equilibrium

The simplest solutions for the gas dynamic equations are those in which the gas is in hydrostatic equilibrium. Even for those simplest solutions, a reduction of the dimensionality is often applied. Hydrostatic equilibrium means that the gas velocity is zero (static) and that the time derivatives are also zero (equilibrium). This implies that if one assumes a barotropic EOS, $p = p(\rho)$, of all the fluid equations one only needs to solve for a simplied form of the momentum equation where the pressure gradient is balanced by a force field

$$\boldsymbol{\nabla} p = \rho \mathbf{g} = -\rho \boldsymbol{\nabla} \Psi \tag{79}$$

Read Chapter 5 up to Section 5.1 to learn about hydrostatic solutions.

The book considers a number of classical cases

1. The isotermal self-gravitating slab. This employs "slab symmetry" which means that one considers an slab which is infinite in two of the three cartesian coordinates. Only one needs to consider one of the cartesian coordinates (z).

2. The isothermal atmosphere. This also employs the slab symmetry, but the gravitational force field is external.

3. Self-gravitating spheres. This employs spherical symmetry. These types of solutions are known as *polytropes* and are useful as simplified models for stars. The differential equation that needs to be solved is called the Lane-Emden equation. With one exception (see Sect. 5.4) we will not consider them in this course.

5.3 Isothermal slabs and atmospheres

Read Sections 5.2 and 5.3 to learn about the isothermal slab and atmosphere solutions.

In one of the exercises we will consider a variation of the isothermal atmosphere, namely an isothermal atmosphere around a spherical body.

5.4 Self-gravitating spheres

While we will not be looking at the general case of self-gravitating spheres, we will look at one case. This case is a simple model of an interstellar cloud which is about to form a star, rather than a model of an already formed star. This solution is known as the *Singular Isothermal Sphere*.
The will assume that the gas is globally isothermal, so $p = A\rho$, where A is a constant (proportional to the temperature). With this expression for the pressure, the condition for hydrostatic equilibrium becomes

$$\frac{A}{\rho}\frac{\partial \rho}{\partial r} = A\frac{\partial \ln \rho}{\partial r} = -\boldsymbol{\nabla}\Psi\,, \tag{80}$$

and the Poisson equation is

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) = 4\pi G\rho\,. \tag{81}$$

We can substitute $\partial\Psi/\partial r$ in the Poisson equation using the hydrostatic equilibrium equation, to give

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \ln \rho}{\partial r}\right) = -\frac{4\pi G}{A}\rho\,. \tag{82}$$

If we assume that the solution for ρ is a power law and that $\rho = 0$ for $r \to \infty$, the solution is

$$\rho(r) = \frac{A}{2\pi G r^2} \tag{83}$$

which is singular for $r = 0$, and therefore called the Singular Isothermal Sphere solution. Even though this solution is singular at $r = 0$ and of infinite mass if one lets the radius go to infinity, it has a finite mass within a specified radius (so the singularity at $r = 0$ does not make it diverge). Also, measurements of so-called dense clouds that are close to forming stars (so-called prestellar cores) often show a $1/r^2$ density distribution, at least over part of the core. So this solution provides a reasonable description of the structure of a large part of those pre-stellar cores and is often use to derive properties of such cores from observations.
Non-singular solutions can be found by changing the boundary conditions, for example by specifying a central density and a constant density beyond a certain radius. These solutions are more realistic as the cloud will have a finite central density and a finite extent. They are known as Bonnor-Ebert spheres. However,

although it is possible to calculate such solutions numerically, it is not possible to write them down in the form of an equation. Figure 5.2 in Clarke & Carswell shows such a Bonnor-Ebert solution, comparing it to the derived density structure of an observed cloud. The best fitting part of the solution is in fact very close to following the $1/r^2$ of the singular isothermal sphere.

5.5 Summary

The main points of this chapter are

- Hydrostatic solutions are static ($\mathbf{u} = 0$) solutions in a gravitational field. The equation to be solved is that of the balance of the pressure gradient with the gravitational acceleration.
- Examples of hydrostatic solutions are the exponential atmosphere and self-gravitating spheres, also known as polytropes.

6 The Perfect Wave

This chapter is about waves. Wave phenomena are important in fluid dynamics as they are the means by which disturbances travel through a gas. We are intimately familiar with wave phenomena in gases, as sound is a wave phenomenon in atmospheric gas. From this we also know by experience that the wave speed is not at all the same as the material speed. Sound travels with about 300 m/s at the surface of the Earth, but this does not mean that this is the wind speed.
As we will see below, and as you probably already know, sound (and thus waves) travel with the so-called sound speed. This means that any disturbance will travel with the speed of sound and the use of the sound speed allows us to estimate how quickly one region can respond to another: there has to be time for sound waves to travel from the one region to the other.

Read the introduction of Chapter 6 in Clarke & Carswell.

6.1 Acoustic waves

To describe how waves travel in a uniform medium, one performs what is known as a *linear perturbation analysis*, where one considers small deviations from an equilibrium solution. We take a medium of constant density (ρ_0), at rest ($\mathbf{u} = 0$) and in pressure equilibrium (at pressure p_0) and study the effect of perturbations in the density, pressure and velocity

$$p = p_0 + \Delta p \tag{84}$$
$$\rho = \rho_0 + \Delta\rho \tag{85}$$
$$\mathbf{u} = \Delta\mathbf{u} \tag{86}$$

When substituting these perturbations into the fluid equations (continuity and momentum equations), only keeping 1st order terms in the perturbed quantities and assuming a barotropic EOS, one obtains the classical wave equation. This shows that perturbations travel as waves through a fluid.
To use the equations:

$$\frac{\partial\rho_0 + \Delta\rho}{\partial t} + \boldsymbol{\nabla}\cdot[(\rho_0 + \Delta\rho)\Delta\mathbf{u}] = 0\,, \tag{87}$$

which can be linearized as

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot\Delta\mathbf{u} = 0\,. \tag{88}$$

Similarly, the momentum equation can be linearized as

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \boldsymbol{\nabla} \Delta p \,. \tag{89}$$

If we now assume that the gas has a barotropic EOS, we can write the pressure as a function of density, and thus

$$\boldsymbol{\nabla} \Delta p = \frac{\mathrm{d}p}{\mathrm{d}\rho} \boldsymbol{\nabla} \Delta \rho \tag{90}$$

We can thus replace Δp with $\Delta \rho$ in Eq. 89 and by taking $\frac{\partial}{\partial t}$ of Eq. 88 minus $\rho \boldsymbol{\nabla} \cdot$ of Eq. 89 we obtain

$$\frac{\partial^2 \Delta \rho}{\partial t^2} = \frac{\mathrm{d}p}{\mathrm{d}\rho} \nabla^2 \Delta \rho \,, \tag{91}$$

which is a wave equation for waves with wave speed $\sqrt{\mathrm{d}p/\mathrm{d}\rho}$. This wave speed is called the *sound speed*, symbol c_s.
The general solution of this equation is

$$\Delta \rho = \Delta \rho_0 \exp(i \mathbf{k} \cdot \mathbf{r} - i \omega t) \,, \tag{92}$$

where $\mathbf{k}$ is the wave number vector (its absolute value k is equal to $2\pi/\lambda$ with λ the wave length) and ω is the angular frequency (equal to $2\pi\nu$ if ν is the frequency). Substituting Eq. 92 into 91 gives that

$$\omega^2 = c_\mathrm{s}^2 |\mathbf{k}|^2 \,, \tag{93}$$

which is known as a *dispersion relation*, giving the relation between the wave number k and the angular frequency ω. It can be used to calculate the phase and group velocities of the waves

$$v_\mathrm{phase} = \frac{\omega}{|\mathbf{k}|} \tag{94}$$

$$\mathbf{v}_\mathrm{phase} = \frac{\mathrm{d}\omega}{\mathrm{d}\mathbf{k}} \,. \tag{95}$$

In the case of sounds waves, the velocities are constant and do not depend on frequency, making sound waves *non-dispersive*.
From the relations above we can also see that

$$c_\mathrm{s}^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \tag{96}$$

from which follows that

$$\Delta p = \Delta\rho_0 c_\mathrm{s}^2 \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)\,, \tag{97}$$

and from Eq. 89 it can be deduced that

$$\Delta\mathbf{u} = \frac{\Delta\rho_0 c_\mathrm{s}^2}{\rho_0\omega}\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) = \frac{\Delta\rho_0}{\rho_0}c_\mathrm{s}\frac{\mathbf{k}}{|k|}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)\,. \tag{98}$$

This last equation shows two things. First, the waves are longitudinal, that is their velocity has the same direction as their wave vectors. Second, the amplitude of the velocity variations is $\frac{\Delta\rho_0}{\rho_0}$ times the propagation speed of the wave, that is the wave travels much faster than individual fluid elements.

> Read Section 6.1 upto the last paragraph on page 66 in Clarke & Carswell to see how the wave equation is found from small perturbations.

Clarke & Carswell correctly point out that if there are variations in the fluid, one should perform the linear perturbation analysis in the Lagrangian frame. However, since we are considering the case of a constant density and pressure medium, the distinction between the Lagrangian and Eulerian frames is not important. If one wants to do the same analysis in a stratified medium, such as an exponential atmosphere, one does need to perform the analysis in the Lagrangian frame. This particular example is presented in Sect. 6.2 of Clarke & Carswell.

6.2 Sound speed

As we showed above, the sound speed depends on how the pressure reacts to density changes, or rather since we assumed a barotropic EOS, on the form of that barotropic EOS. As we saw in Sect. 4.2, there are two main versions of such EOS, namely adiabatic and isothermal. There are thus also two versions of the sound speed the adiabatic

$$c_\mathrm{s} = \sqrt{\gamma\frac{p}{\rho}} = \sqrt{\gamma\frac{k_\mathrm{B}T}{\mu m_\mathrm{H}}}\,, \tag{99}$$

and the isothermal one

$$c_\mathrm{s} = \sqrt{\frac{p}{\rho}} = \sqrt{\frac{k_\mathrm{B}T}{\mu m_\mathrm{H}}}\,, \tag{100}$$

equal to the adiabatic one when one takes $\gamma = 1$. Which one is the relevant one depends on the conditions. If the gas is truly isothermal, that is it can cool on time scales smaller than those associated with the sound waves (ω^{-1}, it is the latter expression for $c_{\rm s}$. At the surface of the Earth this is actually not the case. There is cooling, but it is not efficient enough to make sounds waves isothermal. In this case it is the adiabatic sound speed which should be used. For $\gamma = 7/5$ and appropriate values for T and μ one obtains ~ 330 m s^{-1} as the sound speed on Earth.

Read the rest of Section 6.1 (page 67) on page 66 in Clarke & Carswell to learn about the sound speed.

As was explained in the introduction, the sound speed is important when considering fluid problems, as it connects length and time scales. Therefore it is useful to consider what are typical sound speeds for different astrophysical systems. As was shown in the book, the sound speed depends on the temperature. If we take the adiabatic sound speed for a monatomic hydrogen gas ($\gamma = 5/3$, $\mu = 1$) the numbers are like this

T (K)	$c_{\rm s}$ (km/s)
10	0.29
100	0.91
10^4	9.1
10^6	91
10^8	9.1 10^2

These combinations of temperature and velocity often go together. Systems that have velocities of the order of those listed above will often also contain temperatures of this order. For example, a stellar wind of ~ 1000 km/s will blow a bubble in the surrounding gas which is filled with a hot gas of $\sim 10^8$ K.
Because the sound speed is equivalent to the temperature of the gas, for the isothermal case it can be used to write the relation between pressure and density as follows

$$p = \rho c_{\rm s}^2 \qquad \text{isothermal EOS.} \tag{101}$$

This means that the A that we used in Sect. 5.4 and in Sects. 5.2-3 of Clarke & Carswell (and the K that was used in their Sect. 5.6) actually is equal to $c_{\rm s}^2$.

6.3 Shock waves

The momentum equation is non-linear containing a term proportional to the velocity squared. This means that velocity variations will tend to steepen as the faster moving fluid is transported faster than the slower moving fluid. When the faster moving fluid is overtakes the slower moving fluid, something has to happen as they cannot both exist in the same place. This something is a shock, in which the faster moving fluid almost instantaneously brakes, converting a large part of its kinetic energy into thermal energy.
An alternative way to look at shocks is by considering that all perturbations in a fluid travel with the sound speed. Imagine a wall and a gas flowing towards the wall. If the gas has a velocity lower than the sound speed, small perturbations generated at the wall will travel upstream into the gas and communicate to the gas that there is a wall up ahead. This will allow the flow to smoothly adjust to the existence of the obstruction.
If the fluid is moving supersonically towards the wall, the perturbations travelling still at the sound speed, cannot flow upstream as they are carried along by the fluid faster than they can travel upstream. The supersonically moving flow does not 'know' that there is an obstruction ahead and cannot adjust and will collide with the wall at its full speed. In the collision it will almost instantaneously convert its high velocity to a low one, converting the kinetic energy into internal energy.

Read the introduction of Chapter 7 upto the middle of page 79 in Clarke & Carswell.

6.4 Rankine-Hugoniot relations

Shocks are regions in which the collisions between particles convert kinetic energy into internal energy. This happens in a region which has the size of the mean free path λ between particle collisions. This implies that inside the actual shock region the distribution function of the particles is strongly non-Maxwell-Boltzmann. However, the size of the region is very small compared to the rest of the fluid, namely only about a mean free path. It therefore makes sense to treat it as an infinitely thin transition region or a discontinuity. Using this assumption and the fluid equations it is possible to find the relations between the conditions on either side of the shock. The relations are known as the Rankine-Hugoniot relations.

They can be derived from the fluid equations by realizing that in one (cartesian) dimension and in the absence of an external force and radiative cooling, all five equations can be written as

$$\frac{\partial \mathbf{C}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{102}$$

with

$$\mathbf{C} = \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ E \end{pmatrix} \qquad \text{and} \qquad \mathbf{F} = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_y u_x \\ \rho u_z u_z \\ (E+p) u_x \end{pmatrix}, \tag{103}$$

where $\mathbf{F}$ are the so-called fluxes. If any of the quantities in $\mathbf{C}$ is a region is constant, then the flux into and out of that region should be equal. Therefore in a frame of reference in which the shock is stationary, the flux flowing into the shock should be the same as the flux flowing out of the shock, otherwise there would be either a pile up or an emptying of any of the quantities $\mathbf{C}$. Therefore if "1" refers to the material flowing into the shock region and "2" to material flowing out of the shock region, we obtain (writing u for u_x and for the moment not considering the velocities u_y and u_z):

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \\ \frac{1}{2} u_1^2 + \frac{\gamma p_1/\rho_1}{\gamma - 1} &= \frac{1}{2} u_2^2 + \frac{\gamma p_2/\rho_2}{\gamma - 1}, \end{aligned} \tag{104}$$

where in the last relation we divided out a factor ρu and used that $(\gamma - 1)\mathcal{E} = p/\rho$.

Read Section 7.1 starting in the middle of page 79 in Clarke & Carswell to learn about the Rankine-Hugoniot relations. Note that the contents of the paragraph about the relation between pressure and internal energy (bottom of page 81 and top of page 82) is not specific to shocks and was already described in Sect. 4.6 of these lecture notes.

The conditions before the shock (unshocked gas, ρ_1, u_1, p_1) are known as the pre-shock conditions; the conditions after the shock (shocked gas, ρ_2, u_2, p_2) are known as the post-shock conditions. Note that shocks are *not* reversible so it is easy to tell pre- from post-shock conditions: the density and pressure are always higher in the post-shock region.

6.5 Shock jump conditions

There are various ways to write the shock relations. A very common way (but *not* given in the book) is to express it in terms of the shock's Mach number $\mathcal{M}_1$, given by u_1/c_1 where u_1 is the pre-shock velocity, measured in the shock frame.
The Rankine-Hugoniot conditions can then be manipulated into a form

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{2+(\gamma-1)\mathcal{M}_1^2} = \frac{u_1}{u_2} \tag{105}$$

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}\,. \tag{106}$$

Since the sole parameter determining the magnitude of the jumps in ρ, u and p is $\mathcal{M}_1$, this number is often used to express the strength of the shock, for example as in "a Mach 10 shock". Note that the Mach number is always given as positive, so one should really write $\mathcal{M}_1 = |u_1|/c_1$.
Since $\mathcal{M}_1$ has to be larger than 1 for there to be a shock, one sees that u_1 has to be supersonic, and that the density and pressure go up, and the velocity goes down as the flow passes through the shock. From the relations above one can also see the maximum density ratio of $(\gamma+1)/(\gamma-1)$ for $\mathcal{M}_1 \to \infty$, and that there is no such maximum for the pressure change. Furthermore the post-shock velocity u_2 is subsonic, i.e. less than the sound speed in the post-shock gas.

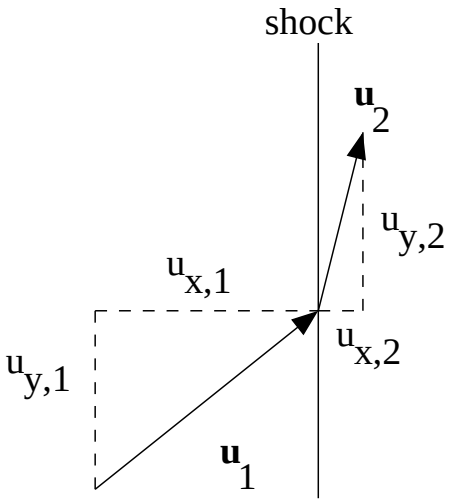

Figure 6: A shock refracting the velocity vector.

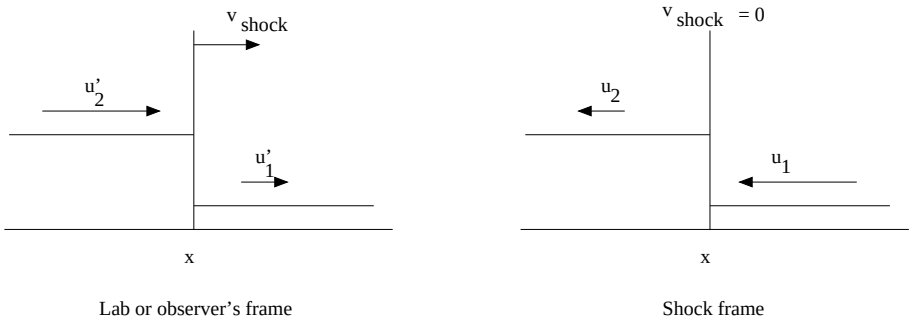

Figure 7: The relation between the observer's and shock frames of reference.

6.6 Refraction across shocks

The velocity component perpendicular to the shock is the one which is reduced across the shock front. The velocity components parallel to a shock do *not* change across the shock, as can be seen from going back to Eq. 103, which for constant mass flux ρu_x (Eq. 105) give that $u_{y,1} = u_{y,2}$ and $u_{z,1} = u_{z,2}$. This means that a shock in the x-direction changes the flow direction (i.e. the velocity vector) if there are y and z components to the velocity vector. This is illustrated in Fig. 6. This is known as shock refraction, as it is reminiscent of the refraction of light rays at when they travel into a medium of different density.
Shock refraction is especially important for the case of curved shocks. Such shocks may have different post-shock conditions as the angle between the pre-shock velocity vector and the shock surface is changing.

6.7 Frames of reference

We derived the Rankine-Hugoniot relations for the so-called *shock frame*, i.e. the frame of reference in which the shock does not move and where the pre-shock material flows into the shock and the post-shock material flows out of the shock. This gives the simplest form for the shock jump conditions. However, one often needs to calculate results in another reference frame, for example that of a star, or a medium at rest into which a shock wave is travelling. This is then called the observer's or lab frame. In this frame the shock may be moving.
Let's call the shock velocity in the lab frame $v_{\rm sh}$, the pre- and post-shock velocities in the lab frame u'_1 and u'_2, and the pre- and post-shock velocities in the shock frame u_1 and u_2. Then obviously, $u_{1,2} = u'_{1,2} - v_{\rm sh}$. See Fig. 7.

Since the shock jump conditions can be conveniently written in terms of the pre-shock Mach number in the shock frame $\mathcal{M}_1$, it is also good to realize that $\mathcal{M}_1 = |u_1' - v_{\rm sh}|/c_1$ (here we dropped the subscript 1 for the Mach number of the again, see Sect. 6.5).
One should also remember that in the lab frame the shock may move in any direction. It could move in direction of the shocked material, and in the direction of the unshocked material. This also means that in the lab frame, the flow on *both* sides of the shock may be supersonic, even though in the shock frame, the pre-shock material has to be super- and the post-shock material sub-sonic. In this sense shock waves are true wave phenomena: they have their own velocity $v_{\rm sh}$ with which they travel.

6.8 Isothermal shocks

If there are no energy losses ($\dot{Q}_{\rm cool} = 0$), the temperature in the post-shock region will be high. Very often this higher temperature will trigger (radiative) cooling processes in the gas and as the gas is moving away from the shock, its temperature will drop (see Fig. 7.4 in Clarke & Carswell). If the cooling processes are so efficient that the temperature returns to the same value it had in the pre-shock region within a distance much smaller than any length in the system, the shock is called an isothermal shock.

Read Section 7.2 in Clarke & Carswell to learn about isothermal shocks.

Isothermal shocks correspond to the case $\gamma = 1$ in the shock jump conditions. This means that much higher compression factors are possible across an isothermal shock.

6.9 Other discontinuities/waves

Apart from shock waves there are other characteristic waves that occur in fluids. The book does not look at these, even though they are quite essential in some astrophysical contexts.

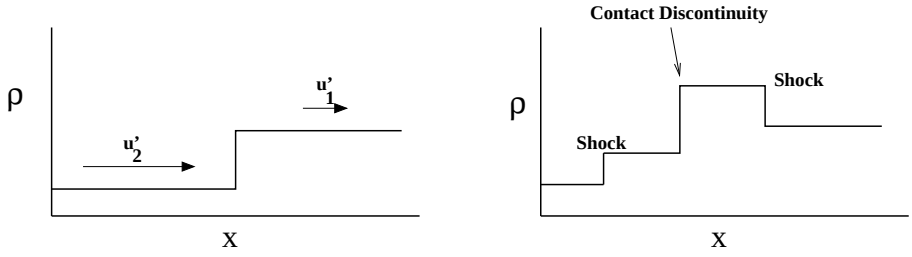

Figure 8: Collision of two flows, leading to the formation of three discontinuities.

6.9.1 Contact discontinuities

Looking back at the Rankine-Hugoniot conditions (Eq.105) one can see that they allow another, seemingly trivial solution:

$$u_1 = u_2 = 0 \quad (107)$$

$$p_1 = p_2 \quad (108)$$

and *no* condition on the densities. This solution does correspond to a physical phenomenon, called a ***contact discontinuity***: a surface without pressure or velocity differences, but *with* a density jump. A contact discontinuity never forms spontaneously, but always originates from an initial discontinuity. Since the pressure is the same on either side, but the density is not, contact discontinuities separate areas of different temperature and entropy. Because of this they are sometimes referred to as entropy waves.
An example of a contact discontinuity is a fast flow hitting a slow flow, as schematically shown in Fig. 8. In this example the fast flow has a lower density. The collision leads to three discontinuities: a shock in the fast flow, a shock in the slow flow, and a contact discontinuity separating the areas of shocked fast and slow flow. The origin of the discontinuity in this case is the initial discontinuity between the fast and slow flows.
Since there is no flow across the contact discontinuities, there are no jump conditions that can be written for them. The jumps in density and temperature can in principle be arbitrarily high.
Contact discontinuities are found in a number of astrophysical problems. One is the interaction of a stellar wind with a surrounding medium. As the stellar wind is pushing the medium aside, there will be a region of shocked stellar wind and a region of shocked environment. These two regions are separated by a contact discontinuity. A similar situation occurs when a supersonic jet is running into a

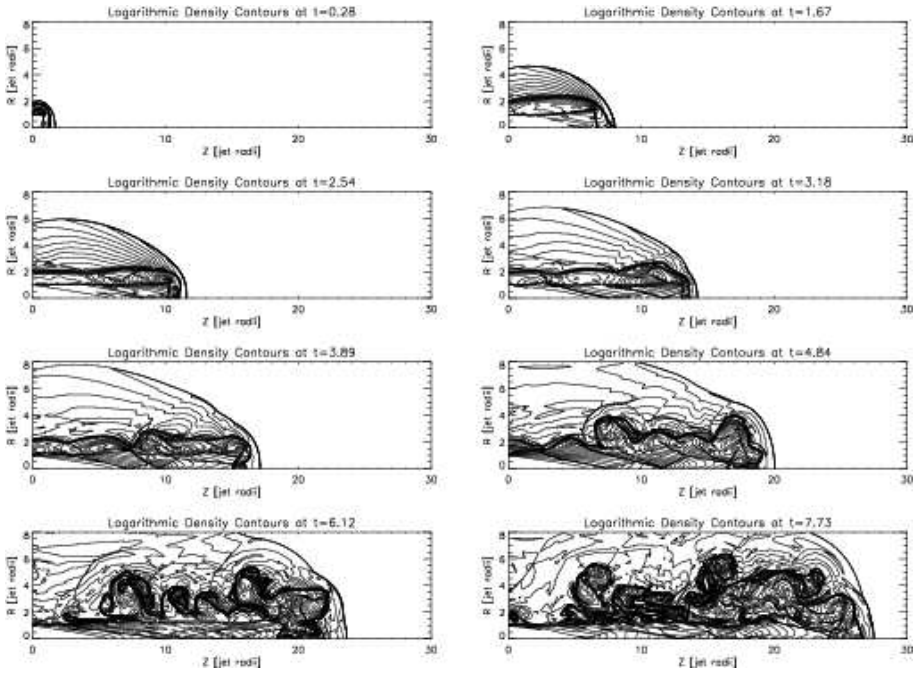

Figure 9: Visualization of a computer simulations of an astrophysical jet from Camenzind & Krause (2001). The figures show density contours. The main jet beam is running along the x-axis (the cylindrical symmetry axis of the problem). The contact discontinuity is mostly clearly visible in the early phases before the flow becomes turbulent. It is the thick pile up of contour lines, indicating a large jump in density. The outermost contours is the so-called bow-shock of the jet, through which the surrounding medium is shocked.

medium, see Fig. 9. You will also encounter contact discontinuities in the colliding flows that are part of the computer lab.

6.9.2 Expansion waves

Shock waves can be said to be compression waves: material gets compressed as it goes through the shock. Contact discontinuities are entropy waves, through which no material travels. The third kind of wave is the so-called ***expansion wave***, which is basically the reverse of a shock wave: material streams in with a high density, low velocity and high pressure, and leaves with a low density, high velocity and low pressure. They are also known as ***rarefaction waves***, as the density is lowered

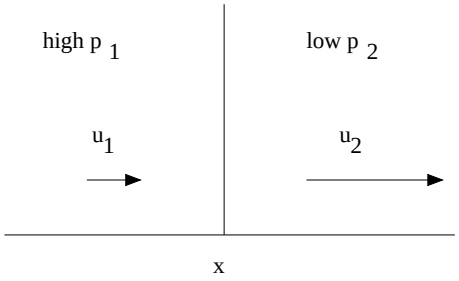

Figure 10: The impossible expansion shock.

when passing through them.
Mathematically, expansion waves could also be discontinuities. However, this would mean that in such an expansion shock internal energy would be converted into bulk kinetic energy, or equivalently, that the entropy would be lowered (see Fig. 10). This goes against the 2nd law of thermodynamics.
As a result expansion waves are not discontinuities, but smooth transition waves, in which both ρ and p change while conserving entropy.
Expansion waves for example occur when you pull out a piston, and the gas has to adjust to the new larger volume (Fig. 11). The head of the wave is the position where the density starts to drop. Since it is a smooth, entropy conserving wave, the velocity of an expansion wave is always the speed of sound c, or if the medium has a velocity $u + c$ or $u - c$.

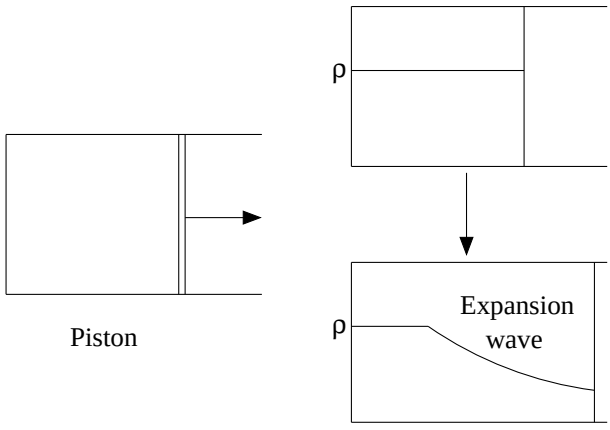

Figure 11: An example of an expansion wave.

6.10 Summary

The most important points from this chapter are

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-
-

7 Blast from the Past

The classical astrophysical example of the use of shock physics is the solution for an idealized explosion, the so-called spherical blast wave, also known as the Sedov-Taylor problem. This solution is relevant for supernova explosions, although not for the very early phases and not for the very late phases.

7.1 Supernova explosions

Supernova explosions are among the most energetic phenomena in the Universe. In a short period energies of around 10^{51} ergs (10^{44} J) are released. Since this amount of energy is released in a relatively short time, the peak luminosity of a supernova can be comparable to that of a whole galaxy.
The origin of the explosion is one of two types of stars.

- **Thermonuclear supernova:** these are the former cores of low to intermediate mass stars ($M_{\rm initial} < 8$ or 9 $\rm M_\odot$) which through nuclear fusion and mass loss have evolved into compact white dwarfs (WD) of mass ~ 1 $\rm M_\odot$ consisting of carbon and oxygen. If such a WD through mass accretion from a companion star reaches the so-called Chandrasekhar mass of 1.4 $\rm M_\odot$ it will start to collapse and initiate carbon fusion in an explosive way which will disrupt the whole star. Observationally these supernovae are classified as type Ia.

- **Core collapse supernova:** these are massive stars ($M_{\rm initial} > 8$ or 9 $\rm M_\odot$) which through different stages of nuclear fusion have built up a central core consisting of elements of the iron group. Nuclear fusion involving elements heavier than iron no longer produces energy, but rather *costs* energy. As the star tries to keep a high pressure in the core to counteract its own gravity, initiating another stage of nuclear fusion beyond iron will lead to the gravitational collapse of the core. Through a serious of processes this collapse is turned into an explosion in which most of the star is ejected, although the core is likely to form a neutron star or black hole. Observationally these supernovae are classified as type II, Ib and Ic (depending on the amounts of hydrogen and helium they show in their spectra).

Supernova explosions are important as they inject a large amount of energy in the interstellar medium of a galaxy, and also distribute heavy elements formed in the star and in the explosion, changing the so-called metallicity (the amount of

elements heavier than helium) of galaxies. The feedback from a supernova may disrupt a star forming cloud, or it may actually trigger further star formation.
A supernova explosion will trigger a shock wave which will travel for a long time ($\sim$ 1 million years) and over large distances ($\sim$ 50 pc) through the interstellar medium. This phase of the expansion of the shock wave, after the supernova itself has faded, is called the supernova remnant. There are many supernova remnants known in the Milky Way. Some from historic supernovae (such as Tycho's and Kepler's from supernova observed on Earth in the 16th and 17th century, and the Crab nebula from the supernova of 1054 AD, recorded by the Chinese), and some from much older supernova (e.g. the Veil nebula from a supernova explosion between 5,000 and 8,000 years ago).

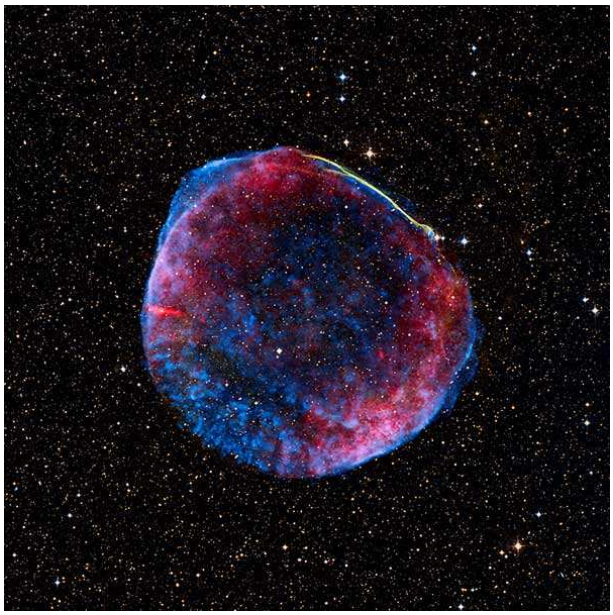

Figure 12: Composite image of the SN 1006 supernova remnant, which is located about 7000 light years from Earth. Blue: X-ray data from Chandra X-ray Observatory. Yellow: optical data from the University of Michigans 0.9 meter Curtis Schmidt telescope at the NSFs Cerro Tololo Inter-American Observatory (CTIO). Orange & lightblue: optical data from the Digitized Sky Survey. Red: radio data from the NRAOs Very Large Array and Green Bank Telescope (VLA/GBT).

The blast wave solution that we will study, has a number of assumptions:

- The explosion only has energy, no mass. This means in practice that it is

only valid for periods in which the mass pushed aside by the explosion is much larger than the mass contained in the explosion. For supernovae this means that the solution is not valid for the initial phases.

- The explosion is spherical and the environment is homogeneous. This means that the solution will be spherically symmetric.
- The pressure of the surrounding medium is negligible compared to the post-shock pressure, implying a strong shock. For supernovae this is not valid at later times, since the shock wave will slow down over time.
- Radiative losses are unimportant. For supernovae this is not valid at later times; as the shock wave slows down, radiative losses become increasinly important.

These assumptions mean a severe simplification of the problem, but it also means that the problem only has two input parameters: the explosion energy E and the density of the environment ρ_0. The fact that it has only those two input parameters leads to an interesting behaviour of the solution, namely that it is self-similar (see below). This solution is called the Sedov-Taylor solution.

7.2 Sedov-Taylor solution

This solution for a strong explosion was found not in the context of astrophysics, but rather in the context of atomic weapons research. Geoffrey Ingram Taylor in Britain and Leonid Ivanovitch Sedov independently discovered the solution. Taylor used his solution to derive the energy released by the first American atomic bomb using only photographs of the explosion published in magazines. This number was considered classified information, so the fact that someone could deduce it from some photographs caused quite some consternation in the US.
The Sedov-Taylor solution consists of two parts. The first part is the expression for how the shock front grows with time:

$$R(t) = \xi_0 \left(\frac{E t^2}{\rho_0} \right)^{\frac{1}{5}} , \tag{109}$$

where ξ_0 is a constant. This part is easy to derive using dimensional arguments.
The second part of the solution describes the internal structure of the blast wave in terms of its density, velocity and pressure profiles. It is these profiles $\rho(r)$,

$u(r)$, $p(r)$ which are self-similar: their shape is the same at all times, apart from a stretching factor (due to the expansion of the blast wave) and an amplitude or scaling factor. So for example, if the solution would be a Gaussian curve, it may widen and the peak value may change, but it would always remain a Gaussian curve.
Written in terms of equations, a similarity solution can be written down as follows

$$S(r,t) = S_1(t)\bar{S}(\xi) \quad (110)$$

$$\xi = rf(t, \text{problem parameters}) \quad (111)$$

where ξ is a dimensionless distance parameter, and f is a combination of the time t and the problem parameters, with dimension 1/length. The function $\bar{S}(\xi)$ is constant in time, but the scale factor $S_1(t)$ is time dependent and the mapping of ξ to the proper distance r is time dependent (the solution gets stretched in case of an expansion).
Clarke & Carswell present the solution in two ways, first in Sect. 8.1.1 with an approximate method where they rely on some ad hoc assumptions and global conservation principles (such as that the change in the total momentum must be equal to the force, that is the pressure times the surface). Then they present the self-similar solution, spread out over two sections, first the expression of how the shock front grows (Eq. 109) in Sect. 8.1.2, then the full similarity solution in Sect. 8.3.

Read Chapter 8 in Clarke & Carswell up to Section 8.4 to learn about the Sedov-Taylor solution.

In Eq. (8.6) the book does the transformation from the shock frame to an observer's frame we described in Sect. 6.7. The post-shock velocity in the observer's frame (u_2' in the notation of Sect. 6.7) is called U in the book.
Please note a few typographic errors in the text of Clarke & Carswell

1. Their equation (8.38) is derived using Eqs. (8.27) and (8.25) according to the text. I suspect they mean Eqs. (8.31) and (8.35).

2. Similarly, their Eq. (8.39) does not follow from Eqs. (8.28) and (8.25), but rather from Eq. (8.36).

3. The p_0 in Eq. (8.39) should be a ρ_0.

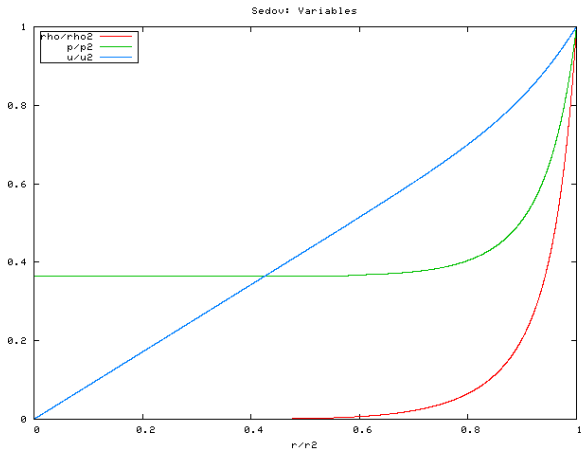

Figure 13: The self-similar Sedov-Taylor solution for the blast wave problem. Red curve: density, blue curve: velocity, green curve: pressure. All curves are normalized to the value at the shock.

The actual form and derivation of the self-similar functions for the density, velocity and pressure (Sect. 8.3 in Clarke & Carswell) are instructive to have seen, but are not exam material. Figure 13 shows the self-similar solutions for ρ, u_r and p for the Sedov-Taylor problem (also shown in Fig. 8.8 in the book). The differences between the three curves are interesting. The density becomes really low in the centre of the blast wave and so most of the material is found near the edge of the structure. The pressure is approximately constant in the centre of the structure and increases near the edge. The ratio of the pressure and density implies that the centre of the blast wave is the hottest part. The velocity is gradually increasing from the centre outward.

7.3 Validity of the Sedov-Taylor solution for supernovae

As also pointed out in the book, the Sedov-Taylor solution is not valid for the early stages of a supernova explosion. This can be seen from the solution for the velocity which diverges for $t \to 0$. Observations show that velocities for the actual explosion are around 10^4 km s^{-1}, so as long as the velocity of the Sedov-Taylor solution is above this, the solution is not valid. Also, the assumption is that the amount of matter swept aside by the supernova explosion should be much more than the mass of the actual explosion ($M_{\rm ejecta}$, the "ejecta"). This means that when

the explosion takes place in a medium of density ρ_0 that

$$R \gg \left(\frac{3M_{\rm ejecta}}{4\pi\rho_0} \right)^{\frac{1}{3}} \tag{112}$$

The ejecta are typically several solar masses for core collapse supernovae and of order a solar mass for a thermonuclear supernova. Using the density value for the book ($\rho_0 = 10^{-21}$ kg m^{-3}) and an explosion energy of 10^{44} J, one finds that the supernova remnant has to be several 1000 years old to reach this condition.
Also for the later stages the Sedov-Taylor solution is not valid. This is described in more detail in Sects. 8.4 and 8.5 of Clarke & Carswell. There are essentially two reasons why the solution becomes invalid:

1. At some point the shock speed becomes of order the sound speed, which is equivalent to saying that the blast wave pressure p_1 becomes similar to the outside pressure p_0. This signals the transition from a shock wave to a normal sonic wave. For the typical parameters used in the book this happens when the blast wave reaches a size of ~ 100 pc.

2. The shocked gas starts to lose significant amounts of energy due to radiative cooling. This is because radiative losses depend strongly on the temperature of the gas. As the blast wave expands, the inside temperature keeps going down. Below a temperature of 10^5 K radiative cooling becomes much more efficient. The Sedov-Taylor solution assumes that radiative cooling can be negelected, so once radiative losses become important the solution changes. For the typical parameters used in the book this happens when the blast wave reaches a size of ~ 20 pc. The new solution gives a slower increase of size of the blast wave $R \propto t^{0.3}$.

In addition the sphericity of the blast wave is more likely to break down once it grows to larger sizes as the blast wave reaches sizes comparable to the thickness of a galactic disk.

Read Section 8.4 and 8.5 in Clarke & Carswell to learn about the breakdown of the Sedov-Taylor solution.

7.4 Summary

The most important points from this chapter are

- A strong explosion into a low pressure environment can be characterized with the explosion energy E and density of the environment ρ_0.
- If these two are the only parameters of the problem, there is no typical length or time scale in the problem, but rather one can construct a unique dimensionless similarity variable $Et^2/(\rho_0 r^5)$ relating length and time.
- From this similarity variable one can directly show that the shock wave will expand as $r \propto t^{\frac{2}{5}}$.
- It is also possible to derive similarity solutions for the density, velocity and pressure which have the same functional form at all times, with time-dependent scaling factors for the amplitude and the radial coordinate.
- The Sedov-Taylor solution is an idealized solution to the problem of a supernova explosion. It is not valid early on (when the mass from the ejects dominates), and it is not valid at later times (when radiative cooling becomes important).

8 Behind the shower curtain

The chapter deals with simple stationary solutions for the gas dynamic equations without shocks. Stationary implies that all the $\partial/\partial t$ terms are zero, but *not* that the gas velocity $\mathbf{u}$ is zero.

8.1 Bernoulli equation

The first stationary 'solution' we consider is the one found already by Daniel Bernoulli in the 18th century. In its full form it describes a quantity in a stationary, barotropic ($p = p(\rho)$) flow that is conserved along stream lines. This quantity is

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi \,, \tag{113}$$

where the term $\int \mathrm{d}p/\rho$ should be interpreted as the indefinite integral of that function. As is shown later in the book our two standard cases of isothermal and polytropic ($p \propto \rho^{1+\frac{1}{n}}$) have the following expressions

$$\left(\int \frac{\mathrm{d}p}{\rho}\right)_{\mathrm{isothermal}} = c_\mathrm{s}^2 \ln \rho \tag{114}$$

$$\left(\int \frac{\mathrm{d}p}{\rho}\right)_{\mathrm{isothermal}} = n c_\mathrm{s}^2 \,, \tag{115}$$

The book uses n here instead of γ to allow for an EOS which has the same form as the adiabatic one ($p \propto \rho^\gamma$), but which does not necessarily have the value for the exponent given by thermodynamics. An example would be a monatomic gas (which has adiabatic index $\gamma = 5/3$) whose temperature through some heating & cooling process is proportional to a power of the density, e.g. $T \propto \rho^{1/3}$. The ideal gas law ($p \propto \rho T$) in this case gives a barotropic EOS $p \propto \rho^{4/3}$, which implies a polytropic index is $n = 3$. However, the γ for this gas is still 5/3, since the intrinsic properties of the gas particles have not changed. So for example the relation between internal energy and pressure is still given by $\rho\mathcal{E} = p/(\gamma - 1) = \frac{3}{2}p$. Because of this it is prudent to separate between n and γ. In the astrophysical literature this difference is often overlooked and γ is used for both (a potential source of confusion).
Note that if the gas is adiabatic so that one can use γ in the EOS, that $\int \mathrm{d}p/\rho = \rho h$, where h is the specific enthalpy introduced in Chapter 4 of these notes.

Read Chapter 9 of Clarke & Carswell up to Eq. (9.10) to learn about Bernoulli's equation.

There are many applications of Bernoulli's equation and the title of this chapter refers to one which is also mentioned in the book.

8.2 Vorticity and potential flow

In deriving Bernoulli's equation the concept of vorticity was introduced:

$$\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}\,. \tag{116}$$

The vorticity is in some cases a convenient quantity to work with. Essentially it is a measure of how much rotation or shear a flow contains. However, one cannot say that a fluid only has vorticity if it rotates: a fluid in solid body rotation does rotate but has zero vorticity. At the same time a fluid with a transverse velocity gradient does not rotate, but does have vorticity. The way to think about vorticity is to take a cross made of two little rods and place it in the fluid. If the rod starts to rotate (change its orientation) as it is being carried along by the fluid, then the fluid at that point has vorticity. To understand this better it is recommended to watch the video "Vorticity, Part 1" from the MIT Fluid Mechanics Films (see the link under `Resources/Extra information` in the Mondo pages).
In many Earth-based fluid problems, the fluid (air or water) can to be approximated as incompressible. In that case the only equation you need to solve is *Helmholtz equation*,

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) \tag{117}$$

(Eq. (9.14) in Clarke & Carswell). For this case, it can also be shown that if an inviscid fluid does not have any vorticity to begin with, it will not develop any (known as Kelvin's vorticity theorem). This is also true for fluids with a barotropic EOS, so for example for an adiabatic flow. This thus opens up for the concept of irrotational fluids which have and keep $\mathbf{w} = 0$. For irrotational fluids one can write the velocity as the gradient of some potential function Φ_u (since $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Phi = 0$ for all Φ). Clearly solving for a potential field Φ_u is easier than solving for a vector $\mathbf{u}$. If the flow vector field can be described by such a potential, one speaks of a *potential flow*.
For the Earth-based problems of incompressible fluids, there is the second condition on the velocity $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. So for an irrotational and incompressible fluid,

one obtains the equation

$$\nabla^2 \Phi_u = 0\,, \tag{118}$$

which is Laplace's equation which one also encounters in other branches of physics such as electrostatics and gravity in vacuum. Many techniques exist for solving this equation.

> Read the rest of Section 9.1 of Clarke & Carswell to learn about vorticity.

8.3 The de Laval nozzle

The second stationary solution we consider is that of a fluid flow through a pipe of varying cross section. This is a famous example from engineering which actually has some relevance for astrophysics. If one constructs a pipe which initially becomes narrower and then widens again, it can be used to smoothly accelerate a fluid to supersonic velocities. This device is known as the "de Laval nozzle" as it was developed by the Swedish engineer and inventor Gustaf de Laval (1845 – 1913), who founded the company which is now known as Alfa Laval. The solution of stationary flow through a pipe of varying cross section shows that the transition to supersonic flow can only happen at the narrowest point of the pipe.

> Read Section 9.2 of Clarke & Carswell to learn about the de Laval nozzle solution.

8.4 Spherical accretion and stellar winds

Perhaps unexpectedly, the idea of the de Laval nozzle that a sonic transition can only happen at a special position (the narrowest part of the pipe in the case of the nozzle), carries over to spherical flows around heavy bodies (stars). Depending on whether the material is falling towards the star or flowing away from it (i.e. the sign of the radial velocity component), we speak of (spherical) accretion or (spherical) stellar winds. In both cases the flow can accelerate to become supersonic, but it can only do this at a very specific point, the so-called sonic point. For a stationary stellar wind the passage through the sonic point is required for the material to be able to escape from the star. This implies that a stellar wind will always become supersonic.

The equation describing a stellar wind (outflow) or spherical accretion (inflow) around a body of mass M is

$$(u^2 - c_{\rm s}^2)\frac{{\rm d}\ln u}{{\rm d}r} = \frac{2c_{\rm s}^2}{r}\left[1 - \frac{GM}{2c_{\rm s}^2 r}\right] \tag{119}$$

which has a critical point at

$$r_{\rm s} = \frac{GM}{2c_{\rm s}^2} \tag{120}$$

Read Sections 9.3 to 9.5 of Clarke & Carswell to learn about spherical accretion and stellar wind solutions.

Stellar winds are common in astrophysics. Our Sun has a stellar wind (of low mass flux) which does not affect the evolution of the Sun much, but which does however have important effects on the planets and comets. Once the Sun evolves into a red giant, it will develop a much more substantial stellar wind which will carry away about about half the Sun's mass within less than 1 million years. Massive stars have stellar winds of high mass flux throughout their lifes.

The two main parameters of a stellar wind are its velocity at large distances from the star (u_∞) and the mass loss rate ($\dot{M}$). For a steady stellar wind the mass loss rate is constant, and since it is the rate of material passing through every radius r, it is related to the density and velocity as

$$\dot{M} = 4\pi r^2 \rho(r) u(r)\,. \tag{121}$$

For large radii, the velocity becomes constant at u_∞, so a steady stellar wind will have a density profile which varies as $1/r^2$.

When a stellar wind interacts with a surrounding medium, a so-called stellar wind bubble is formed. As we have seen a stellar wind will have to become supersonic. This means that it will not be able to react to the presence of a surrounding medium, and a stellar wind bubble will always contain a shock in the stellar wind. If the stellar wind is powerful enough, it will also trigger the formation of a shock in the surrounding medium. There will thus be two shocks, separated by a contact discontinuity. The contact discontinuity separates shocked material from the surrounding medium from shocked stellar wind material, see Fig. 14.

Examples of stellar wind bubbles are so-called Ring Nebulae around massive stars, especially around stars known as Wolf-Rayet stars, which represent a phase of the most intense mass loss in the life of massive stars. Another example are

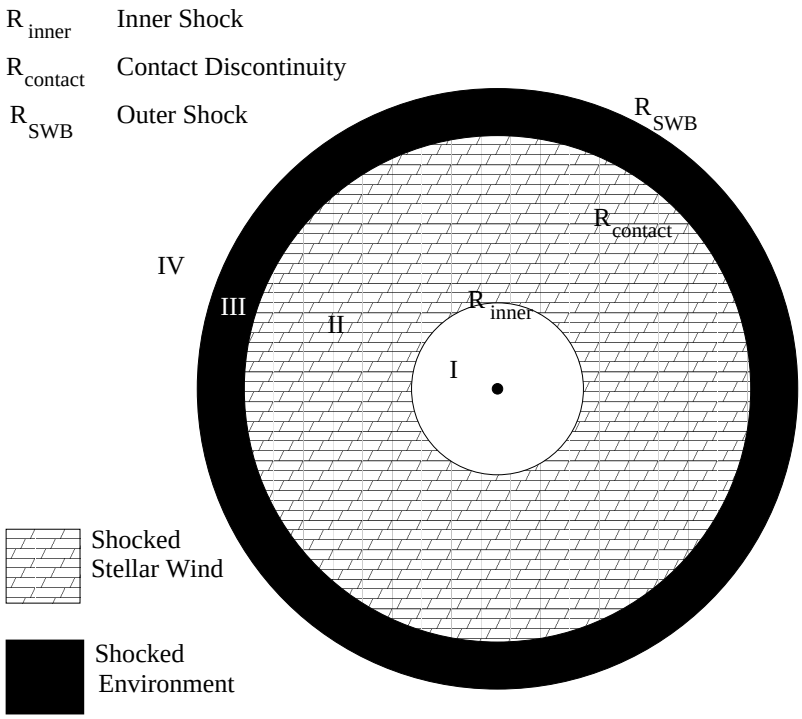

Figure 14: *Stellar Wind Bubble. The inner white area is the unshocked, freely expanding stellar wind (I). The hatched area is the shocked stellar wind (II), the black area is the shocked material from the surrounding medium (III). Outside of that sits the unshocked environment (IV). There are three discontinuities separating these four areas: an inner shock ($R_{\rm inner}$), a contact discontinuity ($R_{\rm c}$), and an outer shock ($R_{\rm SWB}$)*

Planetary Nebulae, which form after low to intermediate mass stars have lost a large part of their mass as a red giant and have turned into hot white dwarfs with fast stellar winds ($\sim$ 2000 km s^{-1}). In both these cases the stellar wind is not actually colliding with the interstellar medium, but with matter previously lost by the star, that is a previous stellar wind phase. Especially Planetary Nebulae are known for their wide variety of shapes, most of them being far from spherical.

8.5 Summary

The following points summarize the most essential concepts introduced in this chapter.

- For stationary flows, the Bernoulli equation gives how the velocity and pressure vary along a streamline: the Bernoulli term H is constant along a streamline.
- The vorticity is defined as $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$. It is a measure of the amount of shear in flows.
- In an inviscid flow, vorticity cannot be generated. Flows without vorticity are called irrotational flows and their velocity field can be described by a scalar potential field; they are therefore known as potential flows.
- The De Laval nozzle with varying cross section A and the equation for stationary spherical accretion or outflow from a body of mass M are both examples of cases where a flow can make a smooth transition from sub- to supersonic flow. However, this transition has to happen at a special location. In the case of the nozzle, at the narrowest point, in the case of a spherical inflow/outflow at the sonic point $\frac{GM}{2c_{\mathrm{s}}^2}$.
- For these flows the conditions at the sonic point determine the solution for the entire problem.

9 Numerical Gas Dynamics: Is your method Good'enuf?

9.1 Introduction

Since analytical solutions to the fluid equations are only possible in a few limited, idealized cases, the development of computers during the second half of the 20th century has led to a whole new branch of fluid studies. In fact, fluid problems were among the first problems to be tackled with computers (during the Manhattan project for the development of the atomic bomb).
Unfortunately, the complexity of the equations also makes them hard to solve numerically, especially in the case of compressible flows containing shocks (as is the usual case in astrophysics). This has led to a large variety of methods for doing computational fluid dynamics (CFD) which all have their limitations.
In this chapter we will look at some of the basic principles which should give you an impression of what CFD is about. The intention of these notes is to illustrate how *physical* principles come back when trying to treat fluid problems numerically.

9.2 Eulerian versus Langrangian

As we saw when studying the basic equations of fluid dynamics, there are two ways to look at a fluid: Eulerian (where one describes the changes in a fluid at given points in space) and Lagrangian (where one follows the evolution of fluid elements). These two ways have their equivalents in numerical methods to study fluid problems. Eulerian methods divide space into a *grid* of points, and solve for the evolution of the density, velocity and pressure at those points. The area around a grid point is called a grid cell. Below we will look at such Eulerian methods.
Lagrangian methods define fluid elements and follow their evolution. As it turns out to be very difficult to follow the evolution of fluid elements of a given initial volume in multiple dimensions (because of the deformation of these volumes due to the flow), the only successful method uses fluid-like particles of a given mass, velocity and pressure and follows these particles as they move around. This technique is called *Smooth Particle Hydrodynamics* or SPH and is widely used in astrophysics. In SPH the local fluid quantities (density, pressure) are defined by smoothing the particles with a so-called smoothing kernel function (hence the term *smooth* particles). The smoothing kernel could for example be a Gaussian. The local density is determined by the number of particles in a given region. One can therefore regard the method as being *adaptive*: particles concentrate in high

density areas and consequently these areas are well sampled. Reversely, underdense regions are less well sampled.
SPH is described more extensively in the book by Michael Thompson (Sec. 6.7, see the document with additional material). SPH works well for self-gravitating flows (where particles tend to collapse to form smaller and smaller structures of higher and higher density), but has problems with large density contrasts and steep gradients such as found in shocks. In fact, the dissipative effects of shocks have to be explicitly added. SPH is very popular in cosmological problems, as it is easily combined with an N-body approach for dealing with the gravity from dark matter. The most widely used package for SPH simulations was developed by Volker Springel and is called *GADGET*.

9.3 Upwind and CFL condition

For reaching some understanding of the techniques used for the Eulerian approach, we start with the advection equation for a constant velocity u, in one dimension (1D). Although relatively simple, this equation owns many of the basic properties of the fluid dynamical equations:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0 \tag{122}$$

To solve this numerically we introduce a discretization (spatial grid and time steps):

$$x_j = x_0 + j\Delta x \tag{123}$$

$$t_n = t_0 + \sum_{n'=0}^{n-1} \Delta t_{n'} \tag{124}$$

If we then specify an initial condition for $\rho(x_j)$ we can follow its evolution by stepping through time with time steps $\Delta t_{n'}$ (which can vary between time steps). For now we take our spatial grid to be uniform: all points are Δx apart. A grid is sometimes also referred to as a mesh; these two words are used interchangibly.
With this discretization we can write several recipes for finding the density at the end of a time step ($\rho(x_j, t_{n+1})$, to be written as ρ_j^{n+1}) from that at the beginning

of the time step (ρ_j^n). For example

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t_n} = -u\frac{\rho_{j+1}^n - \rho_j^n}{\Delta x} \qquad \text{FTFS} \tag{125}$$

$$= -u\frac{\rho_j^n - \rho_{j-1}^n}{\Delta x} \qquad \text{FTBS} \tag{126}$$

$$= -u\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} \qquad \text{FTCS} \tag{127}$$

where FTFS stands for Forward Time, Forward Space, FTBS for Forward Time, Backward Space, and FTCS for Forward Time, Centered Space. These are all *explicit* methods in which the new solution ρ_j^{n+1} can be found directly from the old solution ρ_j^n. If the new solution ρ_j^{n+1} can be found from a combination of the old and the new solution, the method is called *implicit*. In this case one needs to iterate in order to find the new solution.
If one tries these three approaches one finds that

1. FTFS is always unstable for $u > 0$, and sometimes stable for $u < 0$
2. FTBS is always unstable for $u < 0$, and sometimes stable for $u > 0$
3. FTCS is always unstable
4. FTFS for $u < 0$ and FTBS for $u > 0$ are only stable if $\Delta t_n < \Delta x/|u|$

Results for the advection of a square wave using these three methods are shown in Fig. 15, which were taken from the book *Computational Gasdynamics* by Culbert Laney.

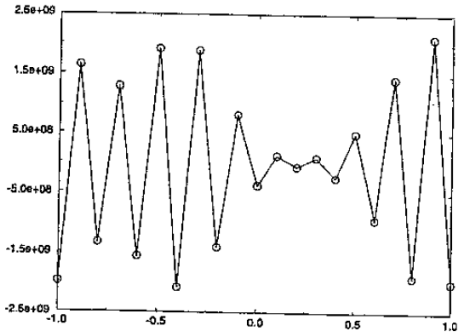

Figure 11.4 Linear advection of a square wave approximated by FTFS.

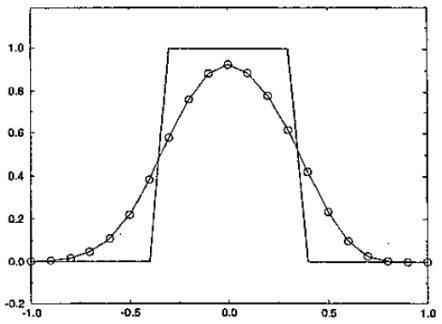

Figure 11.5 Linear advection of a square wave approximated by FTBS.

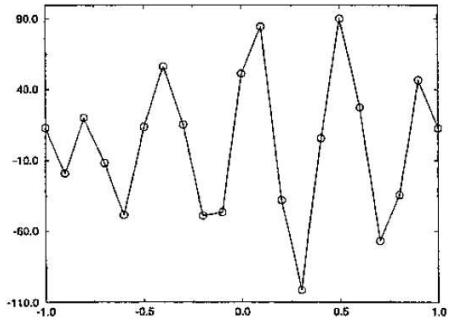

Figure 11.6 Linear advection of a square wave approximated by FTCS.

Figure 15: Figures the behaviour of the FTFS, FTBS and FTCS algorithms for the linear advection equation. The initial condition is a square wave, as shown in solid line in Fig. 11.5. The little circles show the numerical solution after 25 time steps. The advection velocity is 1. Taken from *Computational Gasdynamics* by Culbert Laney.

These results imply that

1. One must take into account the *direction* of the flow (a concept known as *up-wind*).

2. One must choose Δt_n such that one cell only has time to communicates with its direct neighbours. This is of course related to the *speed* of the flow: $\Delta x/|u|$.

Conclusion 2 leads to the so-called Courant-Friedrich-Lewy (CFL) condition

$$\Delta t < \frac{\Delta x}{\max(|a|)} \tag{128}$$

where a are all the flow speeds in the problem. One often writes

$$\Delta t = \eta_{\mathrm{CFL}} \frac{\Delta x}{\max(|a|)} \tag{129}$$

with $\eta_{\mathrm{CFL}} < 1$, the so-called CFL number of the calculation.
For the advection equation $a = u$. However for the Euler equations one also needs to take into account the sound speed, as signals with travel with the sound speed:

$$a = \max(|u - c_s|, |u + c_s|) \tag{130}$$

This connects to the concept of the domain of dependence: only the region of space-time bounded by the lines $u - c_s$ and $u + c_s$ can be reached by the point under consideration (see Fig. 16). If $\Delta t > \Delta t_{\mathrm{CFL}}$, point x_j should physically also affect x_{j+2}, but the methods FTBS and FTFS do not accommodate this. Hence their unstable character in this case.
The CFL condition is only necessary for explicit methods. For implicit methods (in which the new solution is found through iterating over itself), the solution knows about itself through the iteration process and the time step can be larger than given by the CFL condition.
Figure 16 illustrates something else: for the Euler equations there is not a *single* flow direction. Depending on the values of u and c_s, the flow directions may be all positive (needing FTBS), all negative (needing FTFS), or some positive and some negative. Clearly a rather advanced method that can deal with such multiple flow directions is needed for the Euler equations.

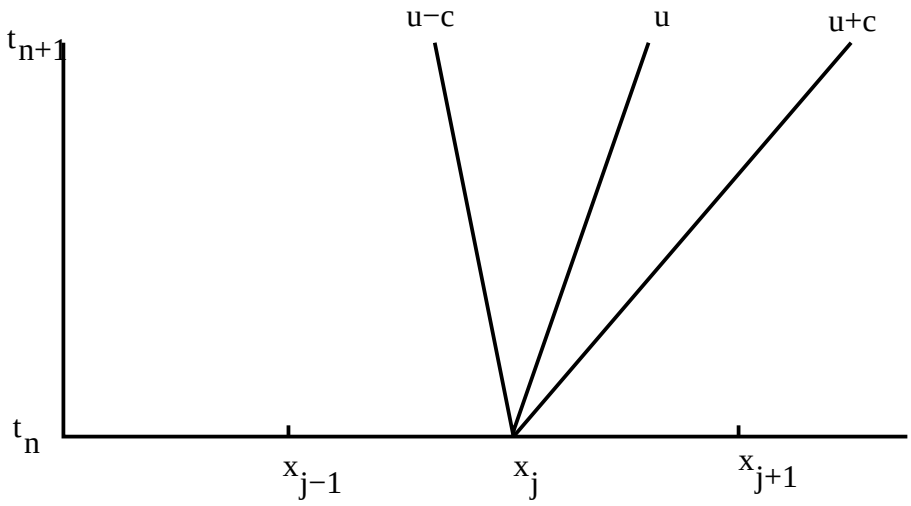

Figure 16: Characteristic directions in space-time delimiting the domain of dependence.

9.4 von Neumann stability analysis

How can we easily analyze the stability of a given method? This can be done using the von Neumann stability analysis (a linear stability analysis technique). Let's consider our advection equation with the FTFS method (Eq. 125), and write ρ_j^n as $C_k^n \exp(ikj\Delta x)$, i.e. a series of sine waves. This is the same approach as used in the linear stability analysis when studying flow instabilities (see Chapter 11), but here it is implemented on a discrete mesh. The idea of the von Neuman method is to evaluate the ratio $R = |C_k^{n+1}|/|C_k^n|$. If R is larger than 1 for all cases, the solution will go to infinity for large n, and is unstable. The von Neumann stability analysis is thus the computational equivalent of the usual linear stabiltiy analysis. Substituting $C_k^n \exp(ikj\Delta x)$ into Eq. 125 gives

$$C_k^{n+1} e^{ikj\Delta x} = C_k^n e^{ikj\Delta x} - \lambda \left(C_k^n e^{ik(j+1)\Delta x} - C_k^n e^{ikj\Delta x} \right) \qquad (131)$$

where we wrote λ for $u\Delta t/\Delta x$. So

$$\begin{aligned} C_k^{n+1} &= C_k^n - \lambda\left(C_k^n e^{ik\Delta x} - C_k^n\right) && (132)\\ \frac{C_k^{n+1}}{C_k^n} &= 1 - \lambda(e^{ik\Delta x} - 1) && (133)\\ &= 1 + \lambda - \lambda\cos(k\Delta x) - i\lambda\sin(k\Delta x) && (134)\\ \left|\frac{C_k^{n+1}}{C_k^n}\right|^2 &= \left(1 + \lambda - \lambda\cos(k\Delta x)\right)^2 + \left(\lambda\sin(k\Delta x)\right)^2 && (135)\\ &= 1 + 2\lambda(1+\lambda)(1-\cos(k\Delta x)) && (136) \end{aligned}$$

which is only always smaller than 1 for $2\lambda(1+\lambda) \leq 0$, or

$$-1 \leq \frac{u\Delta t}{\Delta x} \leq 0 \tag{137}$$

meaning that u has to be negative, and the CFL condition needs to hold, exactly the result given in Sect. 9.3.

9.5 Conservation

Let us look at the Euler equations which for a 1D, cartesian coordinate x can be written as

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{138}$$

where $\mathbf{W} = (\rho, \rho u, E)^T$ is called the state and $\mathbf{F} = (\rho u, \rho u^2 + p, (E+p)u)^T$ the flux of the fluid. A more general case also accomodates for *source terms*

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \tag{139}$$

where $\mathbf{S}$ could be external source terms (force, heating, cooling, etc.), or geometric source terms (due to a non-intertial frame of reference, curvilinear coordinates).
If one defines a cell on our mesh to run from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$ (with the cell centre at x_j), and a time interval from t_n to t_{n+1}, one can write the integral form of the equations as

$$\int_{t_n}^{t_{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial \mathbf{W}}{\partial t}\mathrm{d}x\mathrm{d}t + \int_{t_n}^{t_{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial \mathbf{F}}{\partial x}\mathrm{d}x\mathrm{d}t = \int_{t_n}^{t_{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{S}\mathrm{d}x\mathrm{d}t \tag{140}$$

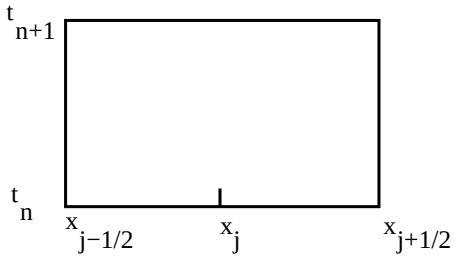

Figure 17: Contour of integration in discretized space-time.

where the double integral is actually a closed curve in space-time (Fig. 17).
This gives

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{W}(x,t_{n+1}) - \mathbf{W}(x,t_n)\mathrm{d}x + \int_{t_n}^{t_{n+1}} \mathbf{F}(x_{j+\frac{1}{2}},t) - \mathbf{F}(x_{j-\frac{1}{2}},t)\mathrm{d}t = \int_{t_n}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{S}\mathrm{d}x\mathrm{d}t \quad (141)$$

Note that we can define the cell averaged state and the time averaged flux as

$$\langle \mathbf{W}(x_j,t)\rangle = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{W}(x,t)\mathrm{d}x \quad (142)$$

$$\hat{\mathbf{F}}(x,t_{n+\frac{1}{2}}) = \frac{1}{\Delta t}\int_{t_n}^{t_n} \mathbf{F}(x,t)\mathrm{d}t \quad (143)$$

and with these defintions we can write

$$\langle \mathbf{W}(x_j,t_{n+1})\rangle = \langle \mathbf{W}(x_j,t_n)\rangle + \frac{\Delta t}{\Delta x}\left[\hat{\mathbf{F}}(x_{j-\frac{1}{2}},t_{n+\frac{1}{2}}) - \hat{\mathbf{F}}(x_{j+\frac{1}{2}},t_{n+\frac{1}{2}})\right] + \Delta t\langle \hat{\mathbf{S}}(x_j,t_{n+\frac{1}{2}})\rangle \quad (144)$$

which hold *exactly*.
For $\mathbf{S} = 0$ this form is known as the *conservative form*, since in this form the conserved quantities $\mathbf{W}$ only change because they receive or give to their neighbours. The spatially integrated values remain constant. See Fig. 18 for an illustrative sketch. The effect of the presence of a source term can be illustrated as in Fig. 19. The conservative form does provide us with a useful general formula for describing a numerical method

$$\mathbf{W}_j^{n+1} = \mathbf{W}_j^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \mathbf{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) + \Delta t\mathbf{S}_j^n \quad (145)$$

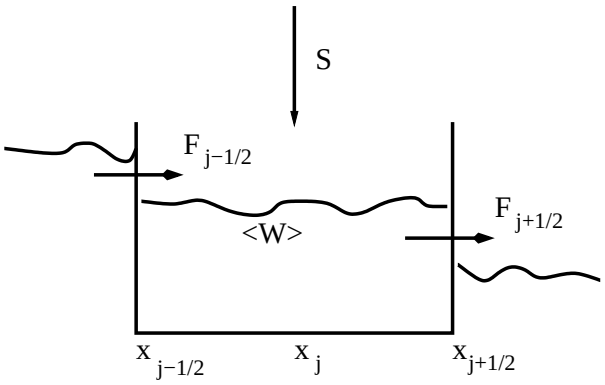

Figure 18: Illustration of the principle of conservative methods.

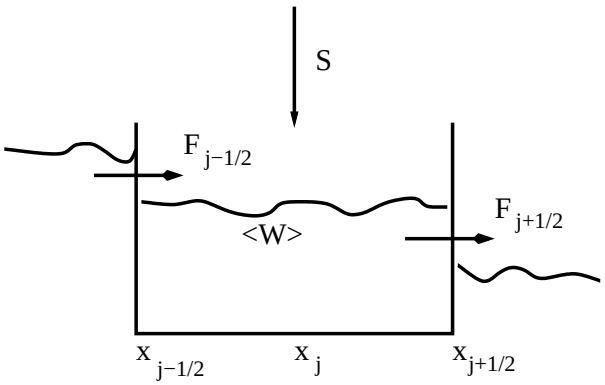

Figure 19: Illustration of the principle of conservative methods with a source term added.

Ignoring $\mathbf{S}$ for now, this formula suggests that our task in designing a numerical hydrodynamics method is about finding a good recipe for $\mathbf{F}^{n+\frac{1}{2}}_{j-\frac{1}{2}}$. Note that this flux in the conservative form is a time-averaged flux

$$\hat{\mathbf{F}}(x_{j+\frac{1}{2}}) = \frac{1}{\Delta t}\int_{t_n}^{t_n} \mathbf{F}(x_{j+\frac{1}{2}}, t)\mathrm{d}t \quad (146)$$

$$\approx \mathbf{F}(x_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) + O(\Delta t^2) \quad (147)$$

$$\approx \mathbf{F}(x_{j+\frac{1}{2}}, t_n) + O(\Delta t) \quad (148)$$

So the more accurate results would be found for a flux value from the intermediate time.

Conservative methods are useful because they automatically conserve the conserved quantities, and the conservative form is also valid at shocks (as we saw when we derived the Rankine-Hugoniot conditions). Since shocks are discontinuities, they are in fact a big challenge for many numerical methods.

9.6 Lax-Wendroff method

We now look at one of the earlier methods developed for CFD. The fact that we need $\mathbf{F}(x_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) = \mathbf{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ suggests the following simple, two step approximation

$$1.\text{Predictor} \quad \mathbf{W}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{W}_j^n + \mathbf{W}_{j+1}^n\right) + \frac{\Delta t}{2\Delta x}\left(\mathbf{F}_j^n - \mathbf{F}_{j+1}^n\right) \quad (149)$$

$$2.\text{Corrector} \quad \mathbf{W}_j^{n+\frac{1}{2}} = \mathbf{W}_j^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \mathbf{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) \quad (150)$$

where $\mathbf{F}_j^n = \mathbf{F}(\mathbf{W}_j^n)$. This method is known as the Lax-Wendroff method, or sometimes as the Richtmeyer method. It works well for smooth flows, but similar to many other early methods gives oscillations near shocks and contact discontinuities. To avoid these the concept of *artificial viscosity* was developed. Artificial viscosity for the Lax-Wendroff method consists of replacing step 2 by

$$\mathbf{W}_j^{n+\frac{1}{2}} = \mathbf{W}_j^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j-\frac{1}{2}}^{n+\frac{1}{2}} - \mathbf{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) + \epsilon\left(\mathbf{W}_{j+1}^n - 2\mathbf{W}_j^n + \mathbf{W}_{j-1}^n\right) \quad (151)$$

where the last term is the artificial viscosity term. It is called artificial viscosity because its form is identical to that of a real diffusion or viscosity, but its coefficient ϵ is chosen purely for numerical reasons. An example of the performance of the Lax-Wendrof method is shown for a shock tube test problem in Figs. 20 and 21 (taken from the book of Laney)[4].
Although methods with artificial viscosity gave reasonable results, many felt uncomfortable with the introducing an arbitrary, tunable and unphysical coefficient ϵ. Note also that the above method ignores the lesson we learned from the advection equation: we should take into account the direction of the flow, or rather (in case of the Euler equations), the sound waves. Method that *do* take this into account are called *upwind methods*.

[4]The shock tube problem is the first problem in the computer lab. Its initial condition consists of two adjacent states one of high pressure and one of low pressure. These develop into an expansion wave going left and a shock wave going right; in between lies a contact discontinuity.

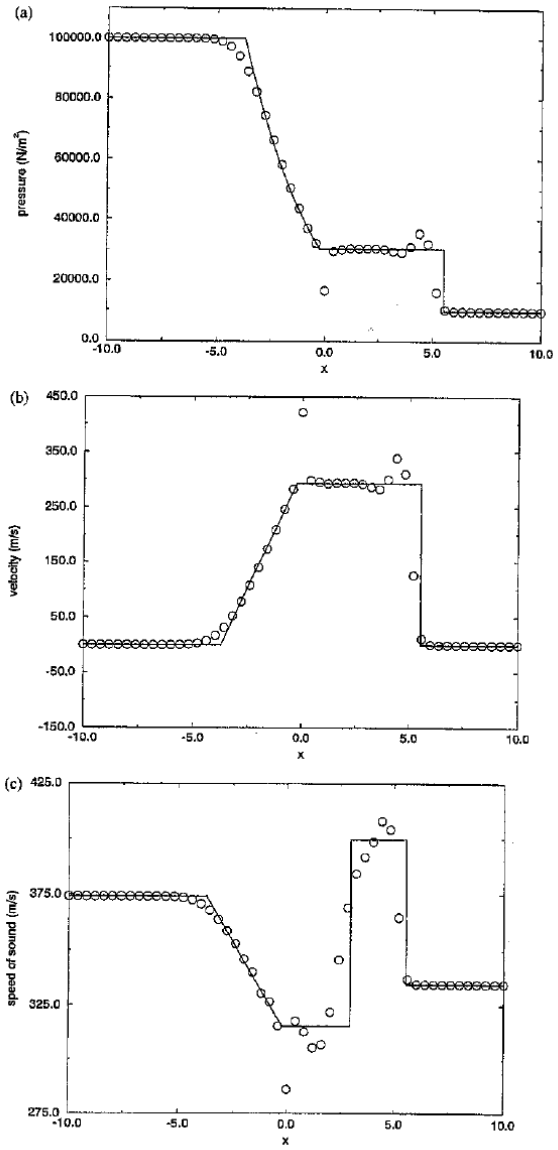

Figure 18.4 The Richtmyer method for Test Case 1.

Figure 20: Results of the standard shock tube test for the Law-Wendroff (or Richtmeyer) method. The boxes show the pressure, velocity and speed of sound. The solid lines are the analytical solution, the circles show the values found by the numerical method. From *Computational Gasdynamics* by Culbert Laney.

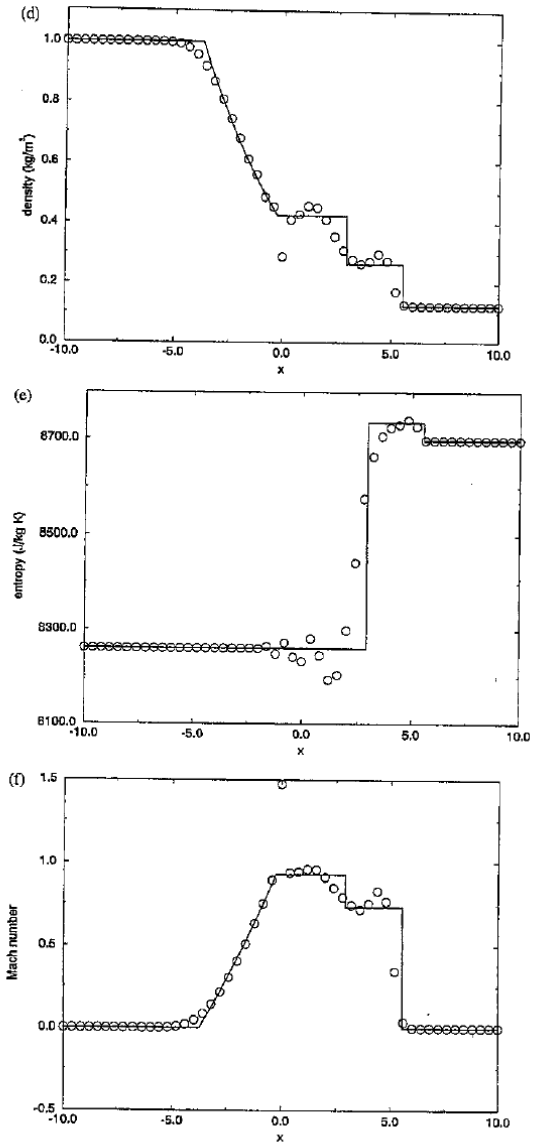

Figure 18.4 (*cont.*)

Figure 21: Results of the standard shock tube test for the Law-Wendroff (or Richtmeyer) method. The boxes show the density, entropy and Mach number. From *Computational Gasdynamics* by Culbert Laney.

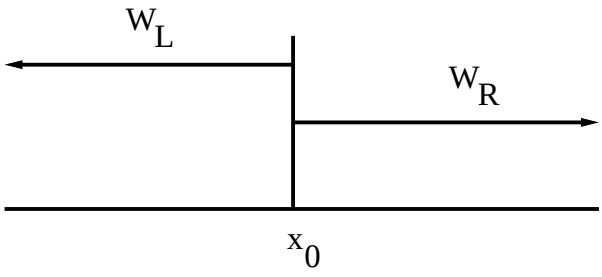

Figure 22: The Riemann problem.

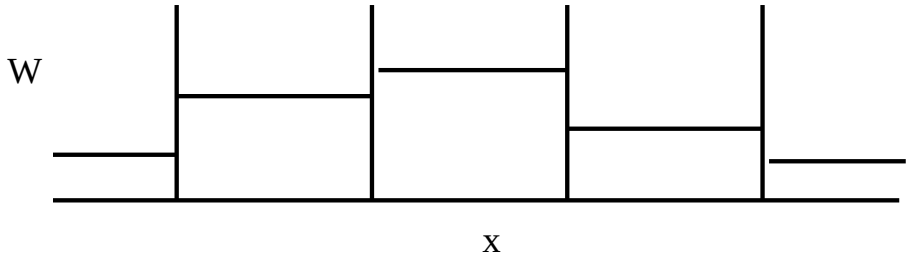

Figure 23: Dividing a mesh into Riemann problems.

9.7 Riemann solvers

A particular class of upwind methods are so-called *Riemann solvers*, or Reconstruction-Evolution methods. This approach, first suggested by Sergei Godunov[5], uses the idea that one can consider the discretized distribution on the mesh as a series of discontinuities. The initial value problem for any discontinuity is known as the Riemann problem,

$$\mathbf{W} = \begin{cases} \mathbf{W}_{\mathrm{L}} & \text{if } x < x_0 \\ \mathbf{W}_{\mathrm{R}} & \text{if } x > x_0 \end{cases} \tag{152}$$

(see Fig. 22) and has known analytical solutions for the Euler equation. If the CFL condition holds, each interface between two cells can be considered to be an isolated Riemann problem (Fig. 23). From the solution of the Riemann problem one finds directly how much mass, momentum and energy flows into a cell from the interface under consideration:

$$\mathbf{F}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \mathbf{F}(W_{\mathrm{Riemann}}(x_{j+\frac{1}{2}}, t_{n+\frac{1}{2}})) \tag{153}$$

[5]The question whether your method is Godunov was the inspiration for the title of this chapter

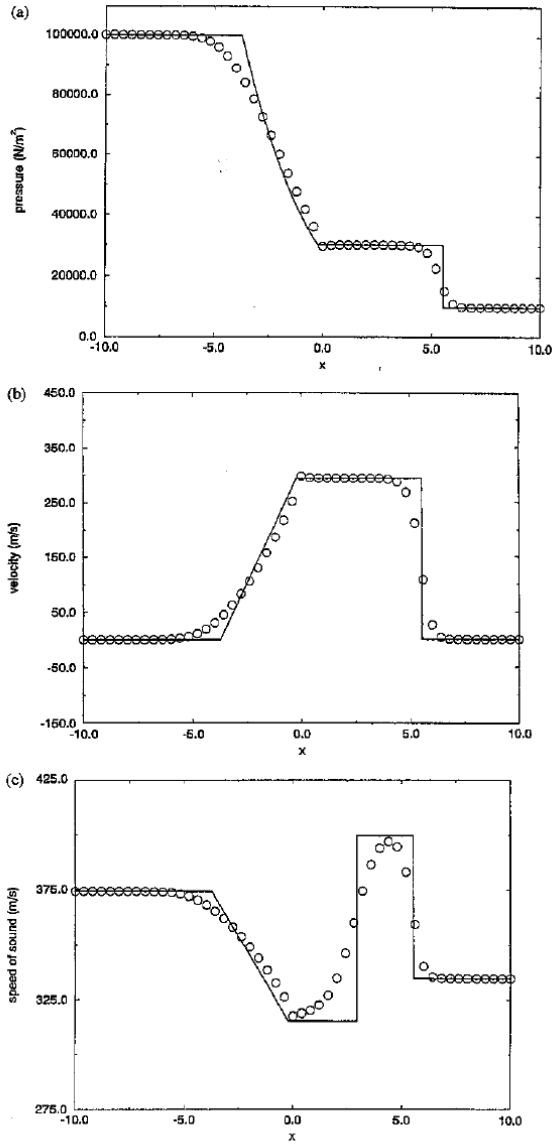

Figure 18.20 Godunov's first-order upwind method for Test Case 1.

Figure 24: Results of the standard shock tube test for the Godunov method. The boxes show the pressure, velocity and speed of sound. From *Computational Gas-dynamics* by Culbert Laney.

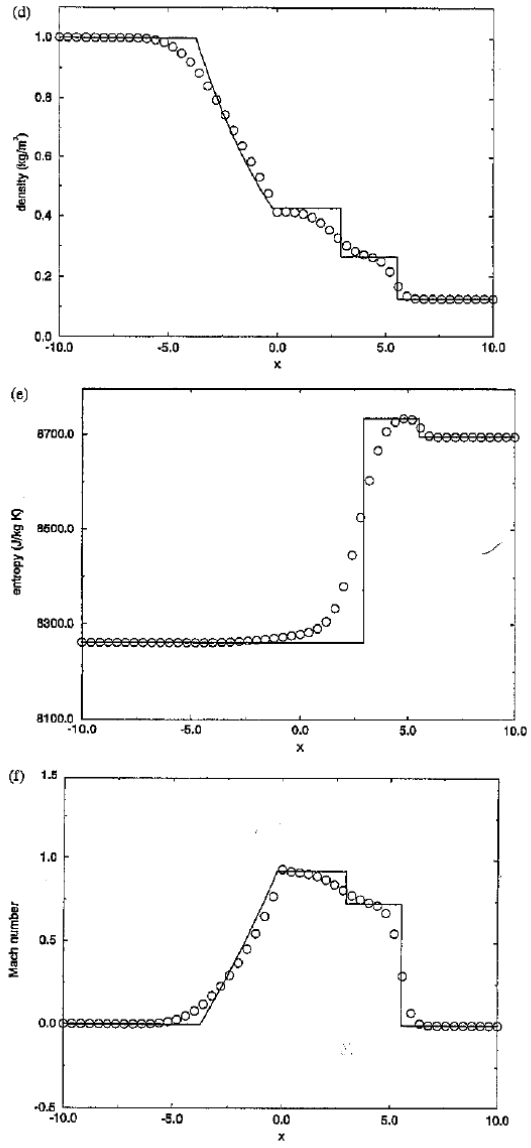

Figure 18.20 (*cont.*)

Figure 25: Results of the standard shock tube test for the Godunov method. The boxes show the density, entropy and Mach number. From *Computational Gasdynamics* by Culbert Laney.

This idea was formulated by Godunov in the 1950s, but was too computationally expensive to implement. In the 1980s computers became fast enough, and also a series of simpler *approximate Riemann solvers* was found. These find an approximate solution to the Riemann problem, good enough to obtain accurate fluxes (which is the only thing needed). The most popular approximate Riemann solver, is the *Roe solver*[6]. A version of the Riemann method popular in astrophysics is known as *PPM* (piecewise parabolic method, Colella & Woodward 1984).
An example of the performance of a Riemann solver (Godunov's upwind method) is shown in Figs. 24 and 25, taken from the book by Laney. Its performance for this test problem is clearly much better than that of the Lax-Wendrof method.

9.8 Multiple dimensions

Up to this point we have only considered 1D methods. How about multi-dimensional methods? In fact, truly multiple dimensional methods are rare. What is most widely used is the technique of operator splitting (or more specifically dimensional splitting).
Suppose that the initial value problem to be solved is

$$\frac{\partial f}{\partial t} = \mathcal{L} f \tag{154}$$

with $\mathcal{L}$ some operator that can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \cdots + \mathcal{L}_m \tag{155}$$

Then the solution can be found by taking steps

$$f^{n+\frac{1}{m}} = U_1(f^n, \Delta t) \tag{156}$$
$$f^{n+\frac{2}{m}} = U_2(f^{n+\frac{1}{m}}, \Delta t) \tag{157}$$
$$f^{n+\frac{3}{m}} = U_3(f^{n+\frac{2}{m}}, \Delta t) \tag{158}$$
$$\text{etc...} \tag{159}$$

where U_k is the solution for $\partial f/\partial t = \mathcal{L}_k f$.
Since the two-dimensional fluid equations can be written as

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \tag{160}$$

[6]This solver is the one used in the computer programme from the computer lab.

we can use the operator splitting approach by first solving for the changes in the x direction, and then using that solution to solve for the changes in the y direction. To avoid systematic effects, the order between these steps can be alternated between time steps. Increased accuracy can be achieved by taking fractional time steps. For two operators:

$$f^{n+\frac{1}{3}} = U_1(f^n, \Delta t/2) \quad (161)$$

$$f^{n+\frac{2}{3}} = U_2(f^{n+\frac{1}{3}}, \Delta t) \quad (162)$$

$$f^{n+1} = U_1(f^{n+\frac{2}{3}}, \Delta t/2) \quad (163)$$

This procedure is sometimes known as *Strang splitting*.

9.9 Public codes

A number of public hydrodynamics codes exist to allow simulations without the need of writing the code yourself. Well known examples are ZEUS, FLASH, NIRVANA and the PENCIL code using grid-based (Eulerian) methods and GADGET for SPH. However, the nature of the numerical solvers is still such that one should be cautious when using them as so-called black boxes, especially for types of problems for which they have not been used before.

9.10 Summary

The following concepts and ideas are the most essential ones from this chapter.

- Numerical methods for the fluid equations can be Eulerian, using grids, or Lagrangian, using particles (SPH).
- The time step for explicit methods is limited by the CFL condition, Eq. 129.
- Numerical methods should take into account the direction of the flow of information in the fluid (the so-called upwind condition).
- The stability of numerical algorithms can be tested with von Neumann stability analysis, the discrete equivalent of the usual stability analysis technique.
- Since the fluid equations are essentially conservation equations, it is important that numerical methods keep conserved quantities conserved. If

numerical algorithms can be formulated using fluxes (Eq. 145), they are conservative.

- Non-upwind methods need an explicit artificial viscosity term to keep them stable.
- An example of an upwind method is the concept of Riemann solvers in which the interface between cells is considered to be a Riemann problem.

10 Transport phenomena: the Honey Trap

10.1 Deviations from Maxwell-Boltzmann

When deriving the Euler equations in Chapter 2 we assumed that the distribution function can locally be described as the equilibrium Maxwell-Boltzmann distribution function $f_{\rm MB}$. However, this is of course a simplification. Even if we have two regions where this locally true, once exposed to each other, collisions between particles that belong to the two different $f_{\rm MB}$ distributions will push the evolution of the combined system towards a new equilibrium $f_{\rm MB}$.
The interaction of particles belonging to different distribution functions gives rise to so-called *transport phenomena*. As the Maxwell-Boltzmann distribution is characterized by the gas velocity $\mathbf{u}$ and temperature T we can expect the additional terms associated with these transport phenomena to have to do with spatial differences in $\mathbf{u}$ and T.

10.2 Chapman-Enskog expansion

We return to the Boltzmann equation, describing the evolution of the distribution function of particles in phase space,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C \,, \tag{164}$$

To proceed further we use the so-called Chapman-Enskog expansion. In this expansion we write the non-equilibrium distribution function f as a series of deviations from the equilibrium Maxwell-Boltzmann distribution ($f^{(0)} = f_{\rm MB}$):

$$f = f^{(0)} + \alpha f^{(1)} + \alpha^2 f^{(2)} + \cdots \quad , \tag{165}$$

where α is a measure of the role of collisions: $\alpha = \lambda / L$ (λ being the mean free path between collisions and L the typical size of our system). For a collisional fluid $L \gg \lambda$ and the α's are small numbers.
Since the α's are small, a first approximation is to write $f \simeq f^{(0)} + \alpha f^{(1)} = f^{(0)} + g$, where we substituted g for $\alpha f^{(1)}$. Putting this back into the Boltzmann equation, and only keeping the first order terms gives

$$\frac{\partial f^{(0)}}{\partial t} + \mathbf{v} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} = C^{(1)}(f^{(0)}, g) \,, \tag{166}$$

where $C^{(1)}(f^{(0)}, g)$ describes the collisions between particles from the equilibrium distribution $f^{(0)}$ and non-equilibrium distribution g.
One can argue that an order of magnitude estimate for the collision term $C^{(1)}$ should be

$$C^{(1)} \approx v_{\mathrm{rel}} \sigma n g \approx \frac{v_{\mathrm{rel}}}{\lambda} g = \frac{g}{\tau} \tag{167}$$

where τ is the typical time between collisions and v_{rel} is the typical relative velocity between colliding particles. This argument suggest the BGK approximation (Bhatnagar, Gross & Krook):

$$\frac{\partial f^{(0)}}{\partial t} + \mathbf{v} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} = -\frac{g}{\tau} \approx -\frac{f - f^{(0)}}{\tau}\,, \tag{168}$$

which implies that the non-equilibrium distribution function f relaxes exponentially to the equilibrium value $f^{(0)}$ with a characteristic time τ.
Since $f^{(0)}$ is a Maxwell-Boltzmann distribution, it only depends on n, $\mathbf{u}$ and T, which all only depend on time t and position $\mathbf{x}$. So we can write that

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{u}}\,, \tag{169}$$

and similar expressions for $\partial f^{(0)}/\partial x_i$.
Putting f_{MB} into Eq. (169) and the equivalent expressions for the spatial derivative, and then substituting these into Eq. (168), gives an expression for g

$$g = -\tau \left[\frac{1}{T}\frac{\partial T}{\partial x_i} w_i \left(\frac{m}{2k_{\mathrm{B}}T} w^2 - \frac{5}{2} \right) + \frac{m}{k_{\mathrm{B}}T} \Lambda_{ij} \left(w_i w_j - \frac{1}{3}\delta_{ij} w^2 \right) \right] f^{(0)} \tag{170}$$

where $\mathbf{w} = \mathbf{v} - \mathbf{u}$, the random velocities of the particles, as before. The term

$$\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{171}$$

is a matrix giving the shear (velocity gradients perpendicular to the velocity directions) in the macroscopic velocity field.

10.3 Macroscopic quantities

We can now derive macroscopic quantities from this first order approximation for the distribution function $f = f^{(0)} + g$. Many terms remain the same as before (when f was $f^{(0)}$) because integrals over terms that are odd in w_i give zero. This is to be expected since the new terms should come from *differences* in the flow.

10.3.1 Heat conduction

We defined the heat flux as

$$\mathbf{q} = \frac{1}{2}\rho\langle w^2\mathbf{w}\rangle \tag{172}$$

which was zero for $f = f^{(0)}$. For g the $\partial T/\partial x_i w_i$ terms result in an integral over w^2, an even term in w_i. Evaluating

$$\mathbf{q} = \frac{\rho}{2n}\int \mathrm{d}\mathbf{w}\mathbf{w}w^2 g \tag{173}$$

gives

$$\mathbf{q} = -K\boldsymbol{\nabla}T \tag{174}$$

$$K = \frac{\tau m}{6T}\int \mathrm{d}\mathbf{w}w^4\left(\frac{m}{2k_{\mathrm{B}}T}w^2 - \frac{5}{2}\right)f^{(0)} = \frac{5}{2}\tau n\frac{k_{\mathrm{B}}^2 T}{m} \tag{175}$$

This is the transport of internal energy due to the existence of a temperature gradient: ***Heat conduction*** or thermal conduction. Clarke & Carswell treated this in Sect. 4.4.2 as a process for energy transport. Here we see that it is in fact a transport phenomenon due to deviations from the Maxwell-Boltzmann distribution.
Heat conduction will strive to diminish temperature gradients, and as quite strong temperature gradients can occur in astrophysical systems, the most extreme ones associated with contact discontinuities, it is potentially an important effect. However, as also mentioned in Sect. 4.4.2 of the book, the role of heat conduction in astrophysics is normally assumed to be quite marginal. The reason for this is that the presence of even a weak magnetic field and some charged particles will reduce the time between collisions τ to such an extent that K becomes very small. The cause of this is that charged particles cannot move freely in the present of a magnetic field and will spiral around the field lines. Therefore their mean free path will be very limited.

10.3.2 Viscosity

In Chapter 2 we defined a tensor $P_{ij} = \rho\langle w_i w_j\rangle$ (Eq. 35). For $f = f^{(0)} = f_{\mathrm{MB}}$ we showed this tensor to be diagonal, and the diagonal elements to be associated with the gas pressure: $P_{ij} = p\delta_{ij}$. For $f = f^{(0)} + g$, the Λ_{ij} velocity shear term in g adds non-zero off-diagonal terms, so we can write $P_{ij} = p\delta_{ij} + \sigma'_{ij}$ where σ'_{ij}

contains all the off-diagonal terms. Working out the integral gives

$$\sigma'_{ij} = m \int \mathrm{d}\mathbf{w} w_i w_j g \tag{176}$$

$$= -\frac{\tau m^2}{k_\mathrm{B} T} \Lambda_{kl} \int \mathrm{d}\mathbf{w} w_i w_j \left(w_k w_l - \frac{1}{3} \delta_{kl} w^2 \right) f^{(0)} , \tag{177}$$

where the suffix notation is implied for the indices k and l.
This $\underline{\sigma'}$ tensor is traceless (this means it has zeros on the diagonal: $\sigma'_{ii} = 0$) and it is symmetric, $\sigma'_{ij} = \sigma'_{ji}$, and proportional to Λ_{kl}. However, Λ_{kl} *does* have diagonal terms since $\Lambda_{kk} = \boldsymbol{\nabla} \cdot \mathbf{u}$, and so not necessarily zero. So to write $\underline{\sigma'}$ in terms of $\underline{\Lambda}$ we need to subtract the divergence of the velocity:

$$\sigma'_{ij} = -2\eta \left(\Lambda_{ij} - \frac{1}{3} \delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{u} \right) \tag{178}$$

where the second term between the brackets makes sure that the total expression is traceless.
The coefficient η must follow from the evaluation of the integral (Eq. 177), for example for σ'_{12}

$$\sigma'_{12} = -\frac{\tau m^2}{k_\mathrm{B} T} \Lambda_{kl} \int \mathrm{d}\mathbf{w} w_1 w_2 \left(w_k w_l - \frac{1}{3} \delta_{kl} w^2 \right) f^{(0)} \tag{179}$$

$$= -2\frac{\tau m^2}{k_\mathrm{B} T} \Lambda_{12} \int \mathrm{d}\mathbf{w} w_1^2 w_2^2 f^{(0)} \tag{180}$$

since the integral is only non-zero when k and l are a combination of 1 and 2. From this we find

$$\eta = \frac{\tau m^2}{k_\mathrm{B} T} \int \mathrm{d}\mathbf{w} w_1^2 w_2^2 f^{(0)} = \tau n k_\mathrm{B} T \tag{181}$$

The tensor σ'_{ij} has to do with non-diagonal terms of Λ_{ij}, so velocity variations perpendicular to the velocity direction, an effect known as *shear*. The property of fluids associated with this is known as ***viscosity***, and σ'_{ij} is known as the *viscous stress tensor*, and η is the viscosity coefficient. Interestingly, the above derivation shows that since $\tau = \lambda / \langle v \rangle$,

$$\eta = \frac{1}{4a^2} \sqrt{\frac{m k_\mathrm{B} T}{\pi}} \tag{182}$$

independent of the density of the gas! This seems counter-intuitive (many would say that "denser fluids are more viscous") but is in fact confirmed by experiments. The reason is that although a denser gas has more particles to transport physical quantities, the mean free path of these particles is shorter, and they are thus less efficient tranporters. Of course this independence of the density is only valid for dilute gases, since we derived our equations under the assumption of a dilute gas. One also sees that $\eta \propto \sqrt{T}$, which is understandable since with higher particle velocity, physical quantities should be transported further. Note that this temperature dependence only holds for gases. As you probably know from personal experience, in liquids the viscosity actually goes down with temperature.

10.4 Navier-Stokes equations

These new effects, conduction and viscosity now have to be added to the fluid equations. The continuity equation does not change. The momentum equation now has a more complicated tensor $\underline{\mathsf{P}}$, namely $P_{ij} = -p\delta_{ij} + \sigma'_{ij}$ and after some re-arranging can be written as

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\boldsymbol{\nabla} \cdot \underline{\mathsf{P}} + \rho \mathbf{a} = -\boldsymbol{\nabla} p + \eta \left[\boldsymbol{\nabla}^2 u + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right] + \rho \mathbf{a} \quad (183)$$

Note that for some special fluids, there is also a so-called *bulk viscosity* ζ which is associated with a $\boldsymbol{\nabla} \cdot \mathbf{u}$ term:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\boldsymbol{\nabla} p + \eta \left[\boldsymbol{\nabla}^2 u + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right] + \zeta \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) + \rho \mathbf{a} \quad (184)$$

This bulk viscosity is associated with diagonal elements for the viscous stress tensor $\underline{\sigma'}$, which do not follow from the ideal, monatomic gas-type approach we used to derive $\underline{\sigma'}$. The bulk viscosity is associated with internal degrees of freedom of the particles in a non-ideal gas, which can be excited or de-excited through volume changes. It is generally unimportant in astrophysical applications.
The energy equation becomes

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E + \underline{\mathsf{P}}) \mathbf{u} - \boldsymbol{\nabla} \cdot (K \boldsymbol{\nabla} T) = \rho \mathbf{u} \cdot \mathbf{a} \quad (185)$$

which by taking the $p\delta_{ij}$ part out of P_{ij} can be written as

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E + p) \mathbf{u} + \boldsymbol{\nabla} \cdot (\underline{\sigma'} \mathbf{u}) - \boldsymbol{\nabla} \cdot (K \boldsymbol{\nabla} T) = \rho \mathbf{u} \cdot \mathbf{a} \quad (186)$$

The $\boldsymbol{\nabla} \cdot (K \boldsymbol{\nabla} T)$ term only acts on the internal energy $\rho\mathcal{E}$, but the complicated $\boldsymbol{\nabla} \cdot (\underline{\sigma}' \mathbf{u})$ term (remember that $\underline{\sigma}'$ is a tensor) has contributions both to the kinetic and the internal energy. Manipulation of the equations shows that the *viscous heating term* is given by

$$2\eta \left[\Lambda_{ij} \Lambda_{ij} - \frac{1}{3} (\boldsymbol{\nabla} \cdot u)^2 \right] \tag{187}$$

and is an energy loss for kinetic energy, and an energy gain for the internal energy. This shows that viscosity is an irreversible, dissipative process through which kinematic energy is turned into internal energy. It is often a small term in the equations.

The new set of equations for ρ, $\mathbf{u}$ and E is called the set of the ***Navier-Stokes Equations***. They are similar to the Euler equations but contain extra terms of *higher* spatial derivatives of the velocity and the temperature. This makes them harder to solve, but also introduces the necessity for more boundary conditions.

10.5 Viscosity

Considering deviations from a local Maxwell-Boltzmann equation gives rise to the viscous terms in the momentum and energy equations. Viscosity is often not very important in astrophysics, and hence it is neglected. In Chapter 12 we will consider the only case where viscosity is essential in astrophysics, namely in accretion discs. But as we will see even there it is not the "molecular" (mircroscopic) viscosity from Eq. 182 which is important, but rather the viscosity is used to describe small scale but macroscopic dissipative processes in the disc.

For many Earth-based applications viscosity *is* important. The reason for this is that we often have to deal with fluids interacting with some solid body. For example the flow around the wing of an airplane, or the flow of a gas or liquid through a pipe. Viscous shear will work to reduce the velocity difference between surface and the flow, so viscosity works like a kind of frictional force. The result is the formation of a "boundary layer". In Section 11.5 Clarke & Carswell calculate the viscous stationary flow through a circular pipe and show that the velocity profile is parabolic with zero velocity at the edge of the pipe and the highest velocity in the middle of the pipe (a solution known as the Coette flow). Without viscosity the flow velocity would be constant across the a cross section of the pipe, but the viscous forces will "brake" the flow near the edge of the pipe until eventually it becomes zero there. So, although viscosity is not very important in "free flow", it

becomes important once your gas flows around a solid body and a boundary layer is created.
The viscous terms in the Navier-Stokes equations contain the viscosity parameter η. As we have seen above, η depends on the temperature of the gas, so in principle it can vary in a fluid. However, for many applications these variations are quite small and the coefficient is assumed to be constant, so that it can be taken out of the differentials. When considering the equation for the velocity $\mathbf{u}$ rather than the momentum density $\rho\mathbf{u}$, a factor η/ρ appears. This is often written as ν and is called the kinematic viscosity.

> Read Sections 11.1 and 11.2 in Clarke & Carswell to learn about viscosity and Navier-Stokes equations.

In Chapter 8 we defined the vorticity of a flow (Eq. 116) and showed that for a barotropic fluid one can write the time evolution of the vorticity with the Helmholtz equation (Eq. (9.14) in Clarke & Carswell). If we now consider the effect of viscosity, the Helmholtz equation becomes

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w}\,, \tag{188}$$

with an extra term $\nu\nabla^2\mathbf{w}$. This extra term invalidates Kelvin's vorticity's theorem: a viscous fluid can create vorticity from nothing, and vorticity can also disappear from a viscous fluid. Only when $\nu = 0$ is vorticity conserved.
The above example of the formation of a boundary layer is a perfect example of this creation of vorticity. The viscous forces set up a shear flow along the boundary of a solid object (pipe, wing, etc.) even if it was not there before; and a shear flow has vorticity.

> Read Section 11.3 in Clarke & Carswell to learn about vorticity and viscosity.

10.6 Reynolds number

When considering the equations for the evolution of the vorticity, one can derive an interesting scaling relation.
We introduce scaling factors for the length L and velocity V such that

$$\mathbf{x} = \mathbf{x}' L, \quad \mathbf{u} = \mathbf{u}' V, \quad t = t' \frac{L}{V}, \quad \mathbf{w} = \mathbf{w}' \frac{V}{L}\,. \tag{189}$$

This makes $\mathbf{x}'$, $\mathbf{u}'$ and $\mathbf{w}'$ dimensionless numbers. We then substitute these relations into Eq. 188 to get a dimensionless version of that equation

$$\frac{\partial \mathbf{w}'}{\partial t'} = \nabla' \times (\mathbf{u}' \times \mathbf{w}') + \frac{1}{\mathcal{R}} \nabla'^2 \mathbf{w}' \,, \tag{190}$$

where $\nabla' = L\nabla$, the scaled version of the spatial differentation and

$$\mathcal{R} = \frac{LV}{\nu} \,, \tag{191}$$

the so-called Reynolds number. This is a dimensionless number which expresses the importance of viscosity.

In the inviscid case $\nu = 0$ so that $1/\mathcal{R} = 0$. Then Eq. 190 does not contain any scale; it is completely scale free. This means that for a given flow pattern, for example the flow around a sphere, if the velocity is increased by a certain factor, the flow pattern will not change (obviously this only holds as long as the flow velocity remains subsonic). Also, the flow around a sphere that is twice the size, will look identical to the flow around the original sphere. There is no typical length or velocity scale in the problem.

However, if there is viscosity, this is no longer true since the term $1/\mathcal{R}$ appears. It is inversely proportional to LV so now the flow does depend on the size and velocity scales; the flow pattern around a two spheres, one of radius R_0 and one of radius $R_1 = R_0/2$ will not be indentical if the flow has the same velocity. However, if for two flows around a geometrically similar object the Reynolds number is identical, the flow pattern will be identical. So the flow pattern around the two spheres *will* be identical if the flow velocity around sphere R_1 is twice that around R_0 (assuming the fluids have the same viscosity).

The other application of the Reynolds number is to estimate the relevance of viscosity. For high Reynolds numbers the viscosity terms are small and do not play a major role in the solution of the flow equations. For small Reynolds numbers the viscosity terms dominate. So low Reynolds number flows are very viscous flows.

We can estimate a typical Reynolds number for astrophysical systems. The kinematic viscosity $\nu = \eta/\rho$, where η is the viscosity of the gas, and ρ its density. Typical values for η of gasses are $\eta \sim 10^{-4}$ g cm^{-1} s^{-1}. For length scales of 1 parsec, velocity scales of 1 km s^{-1} and number densities of order 10^2 cm^{-3} we obtain that $\mathcal{R} \sim 10^5$. This is the reason why viscosity is almost never important in astrophysical problems: the typical Reynolds numbers are extremely high.

The Reynolds number also plays a role when studying turbulence. As we will see in the next chapter, the relative scale of the energy dissipation in turbulent media

scales with $1/\mathcal{R}$. This means that only high Reynolds number flows can become fully turbulent. In practice the development of turbulence requires $\mathcal{R} > 3000$.

10.7 Summary

The most important points from this chapter are

- Deviations from the Maxwell-Boltzmann distribution give rise to transport phenomena
  - Heat conduction, due to the transport of internal energy due to random motions. This term is proportional to the temperature gradient.
  - Viscosity, due to shear flows (velocity gradients perpendicular to the velocity direction), this term is proportional to the gradient of the velocity divergence.
- Heat conduction in astrophysical flows is often inhibited due to a weak and tangled magnetic field.
- Viscosity is mostly unimportant in astrophysical flows due to the absence of strong shear flows (as for example around solid objects).
- The importance of viscosity is expressed by the Reynolds number, $\mathcal{R} = \frac{LV}{\nu}$

11 The Big Mess

In earlier chapters we have considered stationary solutions to the flow equations. However, stationary solutions are only interesting if they are stable. So, in principle one should always investigate the stability of a given stationary solution. If it turns out to be unstable, it will not be a flow solution that will occur in nature since any small disturbances will quickly destroy the flow pattern. Unstable flows will very often evolve into turbulent flows and thus have very different behaviour from what may have been intended.

11.1 Instabilities

The presence of an instability means that a given situation or configuration is very sensitive to small perturbations. If you perturb the configuration by a small amount and it does not return to its original state, but instead keeps moving away from it, the configuration is said to be unstable.
This suggests that the way to analyze the stability of a system is to calculate its response to a small perturbation. This is essentially similar to what we did when we derived the equation for sound waves. One imposes a small pertubation and tracks what happens to it. If the pertubation grows, one has found an instability. If it does not grow, the system is stable. In the case we investigated for sound waves (a fluid of constant density and pressure at rest) we found that it is stable since the sound waves represent small oscillating variations around the equilibrium solution.
Since one can start with very small perturbations, one can do this analysis in the linear regime, which means that you neglect all the terms that contain products of perturbed quantities.
As one can describe any function with a Fourier series, it is also customary to parametrize the perturbations using Fourier components

$$\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t)\right]\ . \tag{192}$$

By substituting this in the linearized equations, one can find a relation between $\mathbf{k}$ and ω, the ***dispersion relation***. From this dispersion relation one can find if a given state is unstable, and one can even find which wave modes grow the fastest.

11.2 Linear stability analysis

The procedure of linear stability analysis is as follows

1. Write down the equations in terms of the background solution which stability you want to test plus a perturbation.

2. Linearize the equations so that only terms linear in the perturbed quantities remain.

3. Insert the Fourier term for the perturbation and finds the relation between k and ω. This is called the dispersion relation.

4. Check whether for real k there are any imaginary solutions for ω. If there are, the solution is unstable since the Fourier term contains $\exp(-i\omega t)$ which becomes $\exp(|\omega|t)$ for imaginary ω, an exponentially increasing term.

This is the short version, also described in Sect. 6.3 in Clarke & Carswell.

11.3 Instabilities in stratified fluids

The most well known gasdynamic instabilities are associated with stratified fluids, that is touching fluids of different densities. The two fluids are assumed to be in pressure equilibrium, so the interface between them is actually a contact discontinuity (see Section 6.9.1).
The two standard cases are

1. Classical Rayleigh-Taylor instability: a heavy fluid on top of a lighter fluid in a gravitational field.

2. Kelvin-Helmholtz instability: two fluids which are sliding past each other with different velocities. No gravity is needed.

There exist other important instabilities which we will not consider here. Some are described in some detail Clarke & Carswell (convection, Jeans instability, thermal instability)
Since both the Rayleigh-Taylor and the Kelvin-Helmholtz instability concern stratified fluids one can actually derive a dispersion relation which is valid for both of them.

Read Section 10.1.2 in Clarke & Carswell up to the derivation of Eq. (10.33), the dispersion relation for stratified fluids.

11.4 Rayleigh–Taylor instability

The true Rayleigh–Taylor instability concerns a heavier fluid resting on top of a lighter fluid. Intuitively one can understand that this is an unstable situation. The reverse situation, with the lighter fluid on top should be stable. Use of the dispersion relation for stratified fluids shows that our intuition is right. If the lightest fluid is on top ($\rho' < \rho$) the dispersion relation has only wave-like solutions, oscillating around the equilibrium solution. These waves are known as surface gravity waves. They differ from sound waves as the wave speeds for surface gravity waves of different wave lengths are different; the waves with the longest wavelengths travel the fastest.

> Read the section on Surface gravity waves in in Clarke & Carswell (page 137).

If the heavier fluid is on top ($\rho' > \rho$), the dispersion relation has imaginary solutions for ω when k is real, which means that the flow is unstable. Since in this case $\omega \propto \sqrt{k}$, the smallest wave length perturbations are the ones which grow the fastest.
It is rather rare that one ends up with a heavier fluid on top of a lighter one in the gravitational field of a star or a planet. However, this instability also applies for flows in which there is an *effective* gravity, that is in accelerating flows. These are much more common. Considering the direction of the effective gravity in an accelerating flow, one can see that a decelerating dense shell travelling into a lower density medium (as is the case in the Sedov–Taylor solution for the expansion of a supernova explosion), is equivalent to a denser fluid sitting on top of a lighter fluid; this configuration is unstable. As can be seen from the images of supernova remnants, the shells do look rather filamentary, indicative of an instability.

> Read the section on Static stratified fluid under gravity in Clarke & Carswell (page 137 & 138).

11.5 Kelvin–Helmholtz instability

If the two fluids are stable in the sense that $\rho' < \rho$, but flow past each other with a relative velocity $U - U'$, the dispersion relation shows that the flow will be unstable for some values of k. This is known as the Kelvin–Helmholtz instability. Also here the most unstable wave lengths are the smallest ones. However, for

the Kelvin–Helmholtz instability above some wave length the waves are actually stable.
The most familiar case of a Kelvin–Helmholtz instability are the condensation trails behind airplanes. One can actually see how the instability develops along the trails. Also in astrophysics the Kelvin–Helmholtz instability is quite common. An example can be seen in Fig. 9 in Chapter 6 of these notes. The contact discontinuity in the backflow region of the jet initially is quite regular but becomes more and more unstable.

Read the section on Kelvin–Helmholtz instability in Clarke & Carswell (page 138).

The linear stability analysis above only tells you that the situation is unstable and gives an indication on which modes are the ones which start the instability. However, as soon as the instability starts to really develop, the linear description brakes down. It for example becomes possible for one instability to trigger another, as the heavier fluid starts sliding into the lighter fluid due to the Rayleigh–Taylor instability, the velocity difference induced by the instability may trigger the Kelvin–Helmholtz instability. Unstable flows can become very complex and often develop into turbulent flows.

11.6 Turbulence

Instabilities and perturbations may ultimately lead to a state of random density, velocity and pressure variations, known as turbulence. Turbulence is important in astrophysics and earth-based applications, but turns out to be extremely difficult to describe. In fact, no general theory for turbulence yet exists.
Since we are talking about random variations, a theory of turbulence has to be statistical. In some sense one can argue that one has to construct another layer of statistics on top of the microscopic picture of a fluid consisting of particles

$$\mathbf{v} = \mathbf{w} + \mathbf{u} = \mathbf{w} + \overline{\mathbf{u}} + \mathbf{u}' \tag{193}$$

where $\mathbf{v}$ are the particle velocities, $\mathbf{w}$ are the random velocities of the particles, $\mathbf{u}$ the gas velocity (mean velocity of the particles), $\overline{\mathbf{u}}$ the average gas velocity in the turbulent flow and $\mathbf{u}'$ the turbulent velocities. Since by construction $\overline{\mathbf{u}'} = 0$, analysis of turbulence is about higher order terms $\overline{\mathbf{u}'\mathbf{u}'}$ (just as it was about w^2 in the statistical treatment of particle velocities). As analysis of the w^2 term led

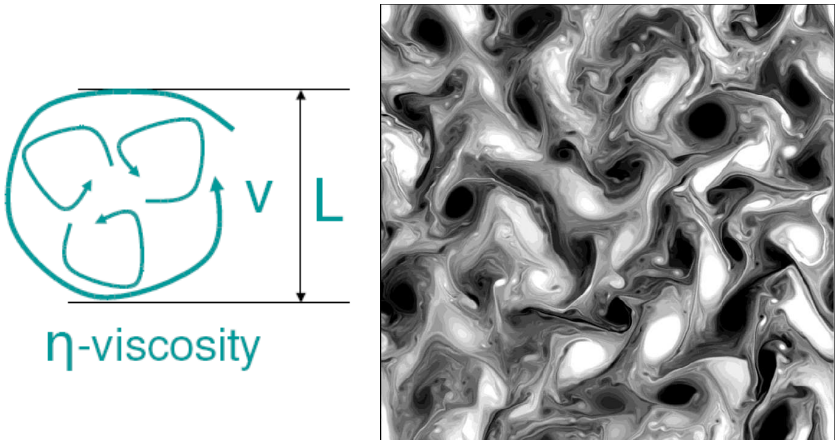

Figure 26: Sketch of a hierarchy of turbulent eddies next to a model calculation of two-dimensional turbulence (Held, I. M., R. T. Pierrehumbert, S. T. Garner, and K. L. Swanson, 1995: Surface quasi-geostrophic dynamics. Journal of Fluid Mechanics, 282, 1).

to the pressure or internal energy of the gas, also turbulence studies focus on the turbulent energy, represented by $\overline{\mathbf{u}'\mathbf{u}'}$.
Unlike the random particle velocities, turbulence has a scale, meaning that turbulent flows have structure. The image often used is that of turbulent 'eddies'. An eddy is swirly or vortex-like flow pattern. Turbulence seems to consist of a hierarchy of bigger eddies containing smaller eddies, that again consist of even smaller eddies, down to the small scales where dissipative processes appear. This image led to the *Kolmogorov* picture of the distribution of turbulent energy over length scales

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{194}$$

where ϵ in the energy input rate per unit mass, and k is the Fourier wavelength, $2\pi/l$, with l the length scale. This relation was derived by Andrey Kolmogorov in 1941 from heuristic and dimensional arguments, but remarkably enough seems to give a fairly accurate description of the energy spectrum of fully developed turbulence.
The dissipation process that removes the turbulent energy from the flow in the smallest eddies are viscous processes. One can therefore use the Reynolds number to analyze a turbulent flow. As we defined before (Sect. 10.6) the Reynolds number $\mathcal{R}$ for a system of size L, velocity V and kinematic viscosity ν is given

by

$$\mathcal{R} = \frac{LV}{\nu} \tag{195}$$

Dissipative processes dominate for length and velocity scales where $\mathcal{R} \sim 1$. For a turbulent hierarchy of eddies this means that $\mathcal{R}$ will be large for the largest eddies ($LV/\nu \gg 1$ if L and V are the size and velocity of the large scale flow), and smaller and smaller for the smaller eddies. At some length scale l_d one reaches the condition $l_d u_d/\nu \sim 1$ and the turbulent energy is dissipated through viscous processes.

To connect the largest scales to the dissipation scales we have to consider the energy cascade from largest to smallest scales. Since energy cannot pile up at a given length scale, there has the be a steady energy rate at all scales. If the energy is fed in at the largest scales L and V at a rate per unit mass ϵ (SI units J s^{-1} kg^{-1}), then a steady flow of energies from large to small scales requires that at every scale l (where the typical velocity is u)

$$\epsilon \sim u^3/l \tag{196}$$

(from dimensional arguments). This implies that $u \sim (\epsilon l)^{1/3}$. At the scale of the system (L, V) this should also hold, defining the input energy rate per mass unit as $\epsilon \sim V^3/L$. At the dissipation scale l_d we have $l_d u_d \sim \nu$, and so $l_d \sim \nu^{3/4}\epsilon^{-1/4}$ and $u_d \sim (\nu\epsilon)^{1/4}$. This then implies that

$$\frac{L}{l_d} \sim \mathcal{R}^{3/4} \tag{197}$$

$$\frac{V}{v_d} \sim \mathcal{R}^{1/4} \tag{198}$$

Where $\mathcal{R}$ is the Reynolds number at the largest scale of the turbulent system (LV/ν). So given the Reynolds number of the system the dissipation scale and velocity can be found. For example a system of large scale Reynolds number 10,000 and size L and turbulent velocity V will have smallest eddies of size $L/10^3$ and typical velocity $V/10$.

To get the energy spectrum $E(k)$, one should realize that that $k \sim l^{-1}$. At scale k the energy is given by

$$E(k)\mathrm{d}k \sim E(k)k \sim u^2 \sim (\epsilon l)^{2/3} \sim (\epsilon/k)^{2/3} \tag{199}$$

which then gives the Kolmogorov energy spectrum, Eq. 194. One can also derive this from dimensional analysis assuming that $E(k)$ only depends on k and ϵ. Obviously, the spectrum will be cut off at small k (large l) because of the size of the

system (L) and at high k (small l) because the dissipation scale is reached (k_d). Experiments for many different systems show the Kolmogorov relation to be valid and surprisingly universal.

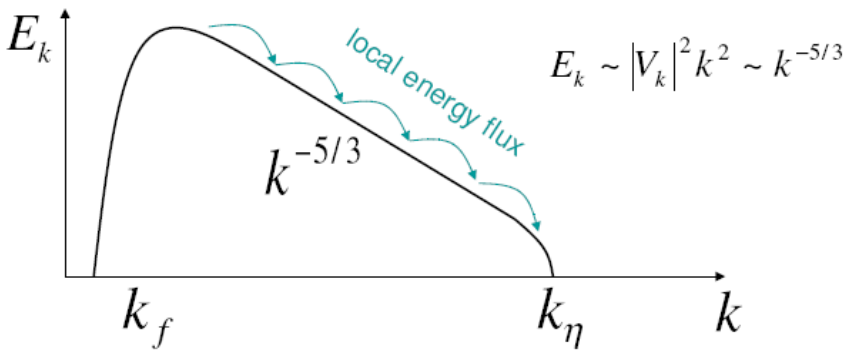

Figure 27: The Kolmogorov energy spectrum for turbulence. The k's corresponding to the size of the system and the dissipation scale are called k_f and k_η respectively.

Since $\mathcal{R}$ is large for most astrophysical systems, turbulence occurs frequently. Some examples are

- turbulent convection in stars
- turbulence in molecular clouds
- turbulent boundary layers around jets
- atmospheric turbulence causing astronomical seeing

11.7 Summary

The most important points from this chapter are

- The stability of a stationary flow pattern can be analyzed using linear perturbation analysis. The key result of such an analysis is the dispersion relation between the wave number k and the frequency ω. If real k's can give imaginary ωs, the flow is unstable.

- The Rayleigh-Taylor instability occurs when a heavier fluid sits on top of a lighter fluid in a gravitational field or in an decelerating flow.
- The Kelvin-Helmholtz instability occurs when a two fluids separated by the contact discontuinity slide over each other.
- Turbulent flows occur in high Reynold number flows and thus are frequent in astrophysical settings. The turbulent energy is typically distributed over the length scales in the flow according to the Kolmogorov spectrum.

12 Accretion Disks

Gravity is the dominant force in astrophysics. When a heavy body such as a star or a black hole starts collecting gaseous material from its environment, we call this process accretion. Usually the accreting material contains some angular momentum, which will cause the accreting material to start to rotate around the accretor. Conservation of angular momentum means that the rotation velocity will be higher the closer the material gets to the accretor. The rotating material will thus collect in a disc-like form around the accretor.
Examples of such accretion discs are

- discs around protostars (so-called protoplanetary discs in which planets are thought to form).
- discs around neutron stars or black holes in binary system. These objects are known as x-ray binaries, as the discs produce copious amounts of x-rays.
- discs around supermassive black holes in the centres of galaxies. If the discs are massive enough, the system is called an Active Galactic Nucleus (AGN).

Gas falls in and settles in a plane defined by the mean angular momentum vector of the gas supply, since shocks and dissipation will dampen motions in the other directions. This will happen on a free fall time, $t_{\rm ff} = (G\rho)^{-\frac{1}{2}}$, usually quite a short time.
Once settled in a rotating discs, the gas will orbit the object, and not fall in, being kept in place by the centrifugal force. The velocities in the disc will thus follow from this balance between gravitational pull and centrifugal force, just as the velocities of the planets orbiting a star

$$\text{Angular velocity} \quad \Omega^2 = \frac{GM}{R^3} \quad \text{Kepler's 3rd law} \tag{200}$$

$$\text{Orbital velocity} \quad u_\phi = \left(\frac{GM}{R}\right)^{\frac{1}{2}} \quad , \tag{201}$$

where R is the cylindrical radius. Now for a gas this is a shear flow, since Ω is a function of R. Without viscocity this would be a stable situation, as the gas would not feel the shear flow. With viscosity the faster rotating inner regions will lose angular momentum to the outer regions. As they do, they will spiral in, and thus some sort of viscosity is needed for a disc to be an *accretion* disc.

Read the beginning of Chapter 12 of Clarke & Carswell up to Section 12.1 to learn about accretion discs and their shear flows.

12.1 Evolution of accretion discs

It is important to realize that the angular momentum density per unit mass L is given by

$$L = Ru_\phi = (GMR)^{\frac{1}{2}} \tag{202}$$

so for equal mass the inner regions have a lower angular momentum than the outer. However, as we saw above, they do have a higher velocity. This difference in the dependence of rotation velocity and angular momentum when orbiting a massive body can confuse one's intuition when considering orbital mechanics problems. For example, if a space vehicle wants to overtake another space vehicle in the same orbit, it should *brake*, not accelerate. By braking it loses angular momentum and moves to a lower orbit where it has a higher velocity with which it can overtake the other space vehicle.

For a gas one can work with the angular momentum density $\mathcal{L}$, given by

$$\mathcal{L} = \rho R u_\phi = \rho (GMR)^{\frac{1}{2}} \, . \tag{203}$$

The evolution of accretion discs is best described by considering the evolution of the angular momentum of the gas. As you should remember from basic mechanics, angular momentum is conserved unless there is a torque acting. The same is true in accretion discs, the angular momentum will be conserved, that is the material will not change its orbit and not accrete, unless there is a torque acting. The torque acting in an accretion disc is the viscous torque from the shear flow.

We will consider thin discs, which means that we can integrate over the z. This means that we will be working with surface densities, rather than volume densities so for the mass:

$$\Sigma = \int \rho \mathrm{d}z \, , \tag{204}$$

and similarly the angular momentum per unit area, $\Lambda = R\Sigma u_\phi$. The book derives an equation for the evolution of Λ (without actually using this symbol) from the equations of gas dynamics in cylindrical coordinates.

Read Section 12.1 of Clarke & Carswell to learn about the angular momentum and surface density evolution in an accretion disc.

12.2 Viscosity in accretion discs

To understand how viscosity makes an accretion disc work, it is useful to derive the solution for the evolution of an initially infinitely thin ring. The viscous forces will spread this ring out. The full mathematical solution (originally found by Jim Pringle) is given in Section 12.2 of Clarke & Carswell. The details of the derivation are not so important, but the result is: the ring will widen with most of the material in the ring moving in; the small amount of material moving out takes with it the angular momentum of the ring (see Fig. 12.3 in Clarke & Carswell). The typical time scale for accretion from an initial radius R_0 is found to be

$$t_{\rm visc} = R_0/\nu \tag{205}$$

also known as the viscous time scale. From this we see that the higher the viscosity the quicker the material accretes.

From observed discs we can obtain an idea of the amount of material accreting to the central object. Since the accretion is driven by the viscosity, this gives us an estimate of value of this viscosity. It turns out that this value is much higher than can be provided by normal molecular viscosity (due to collisions between particles of different distribution functions, as considered in Chapter 10). As is argued in Sect. 12.2.1 of Clarke & Carswell, the Reynolds number of the flow in the disc is enormous, which makes it more or less inevitable that the disc will be turbulent. This means that there will be a lot of turbulent mixing and it is this process which acts as an effective viscosity. One still needs to provide the energy input for the turbulence ("drive the turbulence"), so some process is needed to provide this energy. The currently most popular idea is that this is ultimately a magnetohydrodynamic process called the magnetorotational instability (MRI).

> Read Section 12.2.1 of Clarke & Carswell to learn about viscosity in accretion discs.

As the cause of the viscosity is not really known, a parametrization of the viscosity was suggested by Shakura and Sunyaev (1973):

$$\nu = \alpha c_{\rm s} H \tag{206}$$

where α is a parameter, $c_{\rm s}$ the sounds speed in the disc and H its vertical scale height. This parametrization allows for a handy analytic solution of the structure of the accretion disc, the so-called "α-discs".

12.3 Steady solution

The last thing to consider is a steady solution for the accretion disc. From the steady solution one once again finds that for a given density profile $\Sigma(R)$ the accretion rate is proportional to the viscosity ν. Furthermore one can derive the temperature structure of the accretion disc. As viscosity is a dissipative process, part of the orbital energy lost as the material accretes inward goes into heating the disc. The inner parts will thus generally be hotter than the outer parts. As is shown in Clarke & Carswell if one considers the gravitational potential energy of the accreting material it turns out that half of it is converted into heat and half into kinetic energy. This is in fact another manifestation of the Virial Theorem.

> Read Section 12.3 of Clarke & Carswell to learn about the steady solution and the disc heating due to viscous dissipation.

12.4 Summary

The most important points to remember from this chapter are

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