understand the *physical meaning* of the results obtained. Escape probability methods provide the desired physical insight. As we will see below, they are, for instance, able to assign a clear physical meaning to many of the integral expressions of radiative transfer. Using probabilistic arguments, one is often able to design most suitable approximations to exact expressions.

Second, there is a practical reason. Numerical simulations are capable of providing exact solutions in certain cases, but they can be quite expensive in terms of computer resources. Moreover, they offer high accuracy for solutions of structural equations, but such a high accuracy may not be needed for astrophysical applications, because usually the equations being solved are only approximations to reality. The status of current astrophysical radiative transfer theory is that exact numerical methods are practical for one-dimensional static media, but are either extremely demanding or even completely out of the question for multi-dimensional coupled radiation (magneto)hydrodynamics. Even in one-dimensional simulations, it is worthwhile to have fast numerical methods that allow us to explore wide ranges of parameter space easily, which would otherwise be impossible with detailed numerical methods.

In those situations, it makes sense to use some approximate methods. The most popular and efficient among those are the escape probability methods. The topic has a long history. A comprehensive review of the topic appears in [919]; in this section we summarize its basic concepts. We concentrate on static, one-dimensional media; applications of escape probability ideas to other problems (e.g., moving media) will be discussed in later chapters of this book.

The essence of the escape probability approach is that it provides a simple approximate relation between the radiation intensity and the source function. Having such a relation, one can use it to simplify the problem of coupled radiative transfer equation and kinetic equilibrium equations. Further, it may also provide directly the emergent radiation from the medium. In some cases, the physical meaning of the escape probability methods may be hidden in the formalism, but one should bear in mind that the heart of all escape probability approaches is an approximate relation between intensity and the source function. We first summarize here some results obtained in § 11.8, define the escape probability and related quantities, and then derive some general relations.

Concept of the Net Radiative Bracket

The net rate $R_{ji}^{\rm net}$ for the transition $j \rightarrow i$ is defined by

$$n_j R_{ji}^{\rm net} \equiv n_j A_{ji} + n_j B_{ji} \overline{J}_{ij} - n_i B_{ij} \overline{J}_{ij}. \tag{14.81}$$

The frequency-averaged mean intensity is defined as $\overline{J}_{ij} = \int_0^\infty J_\nu \phi_{ij}(\nu)\, d\nu$; n_i and n_j are the atomic level populations; and A and B are the Einstein coefficients. The first term represents spontaneous emission; the second, stimulated emission; and the third, photoexcitation (absorption of a photon).

It is very useful to express the net rate of the transition between levels j and i as the spontaneous rate times a correction factor,

$$n_j A_{ji} + n_j B_{ji} \overline{J}_{ij} - n_i B_{ij} \overline{J}_{ij} \equiv n_j A_{ji} Z_{ji}, \tag{14.82}$$

where the correction factor Z_{ji} is known as the *net radiative bracket* [1077], the *escape coefficient* [39], or the *flux divergence coefficient* [175].

Noting that the line source function for the transition $i \leftrightarrow j$ is

$$S_{ij} = \frac{n_j A_{ji}}{n_i B_{ij} - n_j B_{ji}}, \tag{14.83}$$

we can rewrite the *net radiative bracket* as

$$Z_{ji} = 1 - (\overline{J}_{ij}/S_{ij}), \tag{14.84}$$

or express $\overline{J}_{ij}$ through Z as

$$\overline{J}_{ij} = (1 - Z_{ji}) S_{ij}. \tag{14.85}$$

The net radiative bracket as such does not immediately help to solve a coupled radiative transfer problem because it depends on the mean intensity, so it can be evaluated only when the solution of the radiative transfer problem is already known. Nevertheless, the concept of the net radiative bracket has utility: suppose that we are able to estimate Z somehow, independently of the radiation field. In this case, we can solve the set of kinetic equilibrium equations for all the atomic level populations, and thus evaluate the line source functions, and finally, compute the radiation intensities by a formal solution of the transfer equation with a known source function. In other words, the difficulties with treating the coupling of radiation and matter (i.e., atomic level populations) would be avoided.

We show below that the escape probability approach is able to provide the desired approximate form of the net radiative bracket.

Concept of Escape Probability

We adopt here a conventional definition of the escape probability to be the probability that a photon *escapes the medium in a single direct flight*, without an intervening interaction with material particles (i.e., without undergoing a scattering process). We note that one may also define a *probability of quantum exit*, which is the probability that a photon will escape the medium directly or after a number of intermediate scatterings [573, 1030]. This kind of escape probability is quite powerful and is often used in analytical radiative transfer (it is essentially a Green function for the problem), but its determination is actually equivalent to a full solution of the transfer problem, so we will not consider this concept any further.

There are different kinds of escape probability, depending on the properties of the initial photon. The *elementary escape probability* is defined for a photon at understand the *physical meaning* of the results obtained. Escape probability methods provide the desired physical insight. As we will see below, they are, for instance, able to assign a clear physical meaning to many of the integral expressions of radiative transfer. Using probabilistic arguments, one is often able to design most suitable approximations to exact expressions.

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a specified position in the medium, with a specified frequency, propagating in a specified direction. Let t_ν be the monochromatic optical depth along the ray from the given point to the boundary of the medium; then the escape probability is given by

$$p_\nu(t_\nu) = e^{-t_\nu}, \tag{14.86}$$

which is equivalent to (11.158).

Consider a plane-parallel, horizontally homogeneous slab. Any ray is specified by its direction cosine μ. In this case the elementary escape probability is both frequency- and angle-dependent:

$$p_{\mu\nu}(\tau_{\mu\nu}) = e^{-\tau_{\mu\nu}}; \tag{14.87}$$

or, writing $\tau_{\mu\nu} = \tau_\nu/\mu$, where τ_ν is the monochromatic optical depth measured inward,

$$p_{\mu\nu}(\tau_\nu) = e^{-\tau_\nu/\mu}, \quad \text{for } \mu > 0, \tag{14.88}$$

and

$$p_{\mu\nu}(\tau_\nu) = e^{-(T_\nu-\tau_\nu)/\mu}, \quad \text{for } \mu < 0, \tag{14.89}$$

because for the opposite direction, the optical distance toward the surface is $(T_\nu - \tau_\nu)/\mu$, where T_ν is the total optical thickness of the slab. Notice that in the case of semi-infinite atmosphere, $T_\nu = \infty$, the escape probability in any inward direction is 0, which is obvious from the basic meaning of "escape."

Averaging over all directions, we obtain the angle-averaged monochromatic escape probability,

$$p_\nu(\tau_\nu) = \tfrac{1}{2}\int_{-1}^{1} p_{\nu\mu}d\mu = \tfrac{1}{2}\int_{-1}^{0} e^{-(T_\nu-\tau_\nu)/\mu}d\mu + \tfrac{1}{2}\int_{0}^{1} e^{-\tau_\nu/\mu}d\mu. \tag{14.90}$$

This equation can be recast into a different form. Taking the second integral (the first one is similar), we write $\int_0^1 e^{-\tau/\mu}d\mu = \int_1^\infty e^{-\tau x}/x^2 dx$. The last integral is the second exponential integral, E_2. Thus we have

$$p_\nu(\tau_\nu) = \tfrac{1}{2}\left[E_2(T_\nu - \tau_\nu) + E_2(\tau_\nu)\right]. \tag{14.91}$$

In the case of a semi-infinite slab, the first term vanishes, and we are left with

$$p_\nu(\tau_\nu) = \tfrac{1}{2}E_2(\tau_\nu). \tag{14.92}$$

These expressions are consistent with those derived in § 11.8. There we introduced the probability, averaged over angles (in one hemisphere), that a photon emitted at $\tau = 0$ will be absorbed in the elementary optical depth range $(\tau, \tau + d\tau)$. This probability is given (see equation 11.162) by $\bar{p}(\tau)d\tau = E_1(\tau)d\tau$. Considering now the photons emitted at optical depth τ in all directions toward the surface at $\tau = 0$, the probability (averaged over angles in the corresponding hemisphere) that such a photon is absorbed between t and $t + dt$ after a direct flight between τ and t is given by $E_1(\tau - t)dt$, and thus the probability that a photon is absorbed anywhere between τ and 0 is $\int_0^\tau E_1(\tau - t)dt = 1 - E_2(\tau)$.

Consequently, the probability that the photon is *not* absorbed between τ and 0, i.e., it *escapes* from the medium, is given by $1 - [1 - E_2(\tau)] = E_2(\tau)$. The probability that the original photon is emitted in a direction toward the surface (as opposed to being emitted into the other hemisphere) is $\frac{1}{2}$, and the final escape probability (in the semi-infinite medium) is given by $\frac{1}{2}E_2(\tau)$, which agrees with (14.92).

It is sometimes convenient to use the *one-sided escape probability*, which represents the probability that a photon emitted isotropically into one hemisphere will escape through the corresponding boundary in a single flight. We denote this probability as $\mathcal{P}_\nu(\tau_\nu)$. In this case, we have

$$\mathcal{P}_\nu(\tau_\nu) = \tfrac{1}{2}E_2(\tau_\nu). \tag{14.93}$$

Finally, we define a *frequency- and angle-averaged escape probability* for an ensemble of photons emitted with probability $\phi(\nu)$:

$$P_e = \int_0^\infty p_\nu \phi(\nu)\, d\nu. \tag{14.94}$$

Consider now the average escape probability for an ensemble of photons in a given line. In this case, $\phi(\nu)$ is the emission profile coefficient. Assume, for simplicity, complete frequency redistribution, in which case the emission profile coefficient is equal to the absorption profile (the more general case where the two profiles may be different is considered in chapter 10). We note that the line absorption profile is normalized to unity, $\int_0^\infty \phi(\nu)d\nu = 1$, and that the monochromatic optical depth in a line is $\tau_\nu = \tau\phi(\nu)$. The one-sided averaged escape probability for line photons is thus given by

$$\mathcal{P}_e(\tau) = \tfrac{1}{2}\int_0^\infty E_2[\tau\phi(\nu)]\phi(\nu)\, d\nu. \tag{14.95}$$

The integral on the right-hand side is an important function of the radiative transfer theory; it is usually denoted as K_2, after [66].

The one-sided averaged line escape probability is thus

$$\mathcal{P}_e(\tau) = \tfrac{1}{2}K_2(\tau). \tag{14.96}$$

Thus the total escape probability for a finite slab is given by

$$P_e(\tau) = \mathcal{P}_e(\tau) + \mathcal{P}_e(T - \tau) = \tfrac{1}{2}K_2(\tau) + \tfrac{1}{2}K_2(T - \tau). \tag{14.97}$$

It is also possible to average the escape probability $P_e(\tau)$ over depth (for finite slabs; the depth average for the semi-infinite medium would be identically zero); this quantity is called the *mean escape probability* or the *escape factor*. These quantities are useful as rough approximations in cases where the medium is treated as a single zone. However, here we are interested in methods that are able to give spatial information, so we do not treat this case.

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The Irons Theorem

The escape probability and the net radiative bracket may be expected to behave in a similar way. Indeed, at large depths ($\tau \gg 1$), we have $J_\nu \to S$, and thus $\bar{J} \to S$. Consequently, $Z \to 0$. The line is said to be in the *detailed radiative balance*. Physically, photons are not able to escape from the large depths; therefore, the total number of radiative transitions $j \to i$ is exactly balanced by the total number of radiative transitions $i \to j$. The escape probability $P_e(\tau)$ also goes to zero for $\tau \gg 1$. Close to the surface, both the net radiative bracket and the escape probability attain their largest values.

Are the escape probability and the net radiative bracket equal at all points in the medium? As we will see below, they are in fact *approximately equal*. However, an interesting *exact relation* also holds, namely, that they are equal in the average sense,

$$\langle Z \rangle = \langle P_e \rangle, \tag{14.98}$$

where the angle brackets denote an emission-weighted (or source-function-weighted) average over the whole volume, i.e.,

$$\langle f \rangle \equiv \frac{\int f(\tau) S(\tau)\, d\tau}{\int S(\tau)\, d\tau}. \tag{14.99}$$

The relation (14.98) is called the *Irons theorem*, because Irons [562] was the first to provide a mathematical proof of what had been a folk theorem for some time. The proof goes as follows. First, one derives a general expression that applies for a single frequency and angle, ν and μ. The emergent intensity along this ray is given by [see equation (11.101)]

$$I_{\mu\nu}(0) = \int_0^\infty S_\nu(\tau_{\mu\nu}) e^{-\tau_{\mu\nu}}\, d\tau_{\mu\nu} = \int_0^\infty S_\nu(\tau_{\mu\nu}) p_{\mu\nu}(\tau_{\mu\nu})\, d\tau_{\mu\nu}. \tag{14.100}$$

This expression has a simple physical interpretation. The term $S_\nu(\tau_{\mu\nu}) d\tau_{\mu\nu}$ represents the number of photons created on the optical depth range $(\tau_{\mu\nu}, \tau_{\mu\nu} + d\tau_{\mu\nu})$, per elementary intervals $d\nu$ and $d\mu$; see § 11.4. This number, multiplied by the escape probability, $p_{\mu\nu}(\tau_{\mu\nu})$, gives the number of emergent photons.

At the same time, the emergent intensity may be obtained by integrating the radiative transfer equation $(dI_{\mu\nu}/d\tau_{\mu\nu}) = S_\nu - I_{\mu\nu}$ without any integrating factor, i.e.,

$$I_{\mu\nu}(0) = \int_0^\infty (S_\nu - I_{\mu\nu})\, d\tau_{\mu\nu}. \tag{14.101}$$

Equating the right-hand sides of (14.100) and (14.101), we obtain

$$\int_0^\infty \left(1 - \frac{I_{\mu\nu}}{S_\nu}\right) S_\nu(\tau_{\mu\nu})\, d\tau_{\mu\nu} = \int_0^\infty p_{\mu\nu}(\tau_{\mu\nu}) S_\nu(\tau_{\mu\nu})\, d\tau_{\mu\nu}, \tag{14.102}$$

or, in the notation of (14.99),

$$\left\langle 1 - \frac{I_{\mu\nu}}{S} \right\rangle = \langle p_{\mu\nu} \rangle. \tag{14.103}$$

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When equation (14.103) is averaged over frequencies and angles, we obtain the Irons theorem.

Physically, the Irons theorem expresses the energy balance of photons in the line. The left-hand side of equation (14.103) represents the excess of the number of emitted photons over the number of absorbed photons, integrated over the whole medium. This number equals the total number of escaping photons, as expressed by the right-hand side.

Escape Probability Treatments

The fact that the escape probability and the net radiative bracket are equal in an averaged sense does not mean that they should be equal locally, at every point in the radiating slab. However, one finds that although the detailed equality of the frequency-averaged mean intensity and the net radiative bracket does not hold *generally*, it is nevertheless a satisfactory *approximation*.

The formal solution for the mean intensity of radiation is

$$J_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(t) E_1(\tau_\nu - t)\, dt + \int_{\tau_\nu}^{T_\nu} S_\nu(t) E_1(t - \tau_\nu)\, dt. \tag{14.104}$$

As discussed in § 11.5, the kernel $E_1(t)$ has a width of the order of one optical depth unit. In contrast, the scale of depth variation of the source function $S(t)$ may be much larger. If we assume that the source function is constant over the region where the kernel E_1 contributes significantly to the integral, then the source function can be taken out of the integral, setting

$$S_\nu(t) = S_\nu(\tau_\nu). \tag{14.105}$$

Equation (14.104) is then modified to read

$$\begin{aligned} J_\nu(\tau_\nu) &= \left[1 - \tfrac{1}{2}E_2(\tau_\nu) - \tfrac{1}{2}E_2(T_\nu - \tau_\nu)\right] S_\nu(\tau_\nu) \\ &= [1 - p_\nu(\tau_\nu)] S_\nu(\tau_\nu). \end{aligned} \tag{14.106}$$

Integrating equation (14.106) over frequencies with weighting factor $\phi(\nu)$, and assuming that the source function is independent of frequency (i.e., the case of a single line with a complete redistribution), we obtain

$$\bar{J}(\tau) = \left[1 - \tfrac{1}{2}K_2(T_\nu - \tau) - \tfrac{1}{2}K_2(\tau)\right] S(\tau) = [1 - P_e(\tau)]\, S(\tau). \tag{14.107}$$

In this case, the net radiative radiative bracket is equal, at all points in the medium, to the escape probability,

$$Z(\tau) = P_e(\tau), \tag{14.108}$$

which can be seen by comparing equations (14.85) and (14.107).

This approximation is called the *first-order escape probability* method. Its computational advantage is immediately clear: If we write the kinetic equilibrium equations in terms of net rates, we may replace Z by the escape probability P_e for all transitions. The rate equations no longer contain an unknown radiation field, so they can be solved easily. Nevertheless, they still must be solved by iteration because the escape probabilities depend on optical depths, which in turn depend on the level populations. But as these iterations are not related to consecutive photon scattering, the iteration process is quite different from the Λ-iteration scheme and is typically much faster.

The derivation above is a purely mathematical one. The only physical point there is the argument concerning the scale of variation of the source function and the kernel function. Therefore, it is useful to examine a more physical derivation, which could shed more light on the nature and limitations of the escape probability method.

Consider first a limiting case of very large optical depth in a medium with constant (or very slowly varying) properties. In this case, the variation of the source function with depth arises only because of the presence of a boundary or boundaries. As discussed above, deep in the medium the escape probability is essentially zero. On the microscopic level, every downward radiative transition is immediately balanced by the upward transition. The balancing transition does not necessarily occur at the same point in the medium, since a photon will travel a distance of the order of one unit of monochromatic optical depth. Nevertheless, because the properties of the medium do not vary over the mean free path of the photon, the resulting picture is the same as if *every emitted photon is immediately re-absorbed at the same point* in the medium. This is the reason why this approximation was historically called the *on-the-spot approximation*; it is also sometimes called *complete line saturation* or, perhaps most frequently, *detailed radiative balance*. In this case,

$$n_j A_{ji} + n_j B_{ji} \overline{J}_{ij} - n_i B_{ij} \overline{J}_{ij} = 0; \tag{14.109}$$

thus

$$\overline{J}_{ij} = S_{ij}. \tag{14.110}$$

Therefore, in this approximation,

$$Z_{ij} = 0, \tag{14.111}$$

and inasmuch as $P_e \approx 0$, here we again have the case where $Z \approx P_e$.

A more general case is provided by the so-called *dichotomous model*. Instead of assuming that all emitted photons are re-absorbed on the spot, we divide them into two groups. The first group of photons is indeed re-absorbed on the spot, while in contrast the rest of photons escape the medium altogether. This is of course an approximation; in reality there is a continuous distribution of distances that a newly created photon can travel, ranging from zero all the way to the optical distance toward the boundary. The dichotomous model essentially approximates the real distribution by a bi-modal distribution. This procedure may seem rather crude, but, as we will see below, it reflects the basic physics of line transfer.

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The fraction of photons that do escape is given by the escape probability. The net rate in the transition, i.e., a difference between the downward and upward transition rate and therefore a difference between the number of photons created and number of those destroyed, is given by the fraction of the spontaneous emission rate that produces the escaping photons, i.e.,

$$n_jA_{ji} + n_jB_{ji}\overline{J}_{ij} - n_iB_{ij}\overline{J}_{ij} = n_jA_{ji}P_e. \quad (14.112)$$

Comparing this equation to (14.82), we see that

$$Z = P_e, \quad (14.113)$$

i.e., the equality of the net radiative bracket and the escape probability is *exact* here. This model is also called the *normalized on-the-spot* approximation, after [1106].

We will see later (chapter 19) that the first-order escape probability (i.e., dichotomous model) is an excellent approximation for media having a large velocity gradient; the approximation is called there the *Sobolev approximation*. In static media, the situation is more complex. We return to this point later.

Finally, we give asymptotic expressions for the one-sided escape probability as $\tau \to \infty$ when the absorption profile coefficient $\phi(\nu)$ is given by Doppler and Voigt profiles; see [66]:

$$P^D(\tau) \approx \frac{1}{4\tau\sqrt{\ln(\tau/\pi^{\frac{1}{2}})}}, \quad (14.114a)$$

$$P^V(\tau) \approx a^{\frac{1}{2}}/3\tau^{\frac{1}{2}}. \quad (14.114b)$$

The approximate expressions derived using the Osterbrock picture (14.30) agree with the exact asymptotic expression (14.114) within a factor of 2 for the Doppler profile and a factor of $\sqrt{\pi}/3$ for the Voigt profile. Thus despite the approximate nature of the Osterbrock picture, its results are in surprisingly good agreement with exact asymptotic results. This shows that it provides a valuable insight into the nature of the escape probability and enables us to derive some basic expressions very easily.

Core-Saturation Method

We have seen above that the idea of a core-wing separation of line photons enables us to calculate an approximate escape probability very easily. In his *core-saturation method*, Rybicki [918] showed that this approach can be used not only to evaluate escape probabilities, but also to treat the entire radiative transfer process.

Consider a beam of radiation along a ray in a plane-parallel, horizontally homogeneous medium. Let τ_ν be the monochromatic optical depth along this ray measured inward from the boundary. The frequency dependence of the opacity and therefore τ_ν may be arbitrary; for the sake of simplicity we assume here radiative transfer in a single line, in which case $\tau_\nu = \phi_\nu \tau$, where τ is the frequency-averaged optical depth in the line, and ϕ_ν is its absorption profile coefficient.

At any depth in the medium, divide frequency space into two parts, the *core* and the *wing*. To this end, choose a parameter $\gamma \approx 1$ such that in the core region, $\tau_\nu \geq \gamma$. Then make the approximation that

$$I_\nu = S_\nu \quad \text{for} \quad \tau_\nu \geq \gamma. \tag{14.115}$$

The remaining part of frequency space, defined by $\tau_\nu < \gamma$, is the *wing* region. In this region, we do not impose any approximation on S_ν. The core-wing separation is dependent on the position in the medium. The division frequency between the core and the wing region, x_d, is given by

$$\tau\phi(x_\mathrm{d}) = \gamma. \tag{14.116}$$

For instance, for a Doppler profile we have, in analogy to equation (14.29a),

$$x_\mathrm{d} = \sqrt{\ln(\tau/\gamma\pi^{\frac{1}{2}})}. \tag{14.117}$$

The next step is to write down the corresponding expression for the net radiative rate. First, we express the frequency-averaged mean intensity $\overline{J}$ as

$$\overline{J} = 2\int_0^\infty J_x\phi(x)\,dx = 2\int_0^{x_\mathrm{d}} J_x\phi(x)\,dx + 2\int_{x_\mathrm{d}}^\infty J_x\phi(x)\,dx \equiv \overline{J}_\mathrm{c} + \overline{J}_\mathrm{w}, \tag{14.118}$$

i.e., we split the frequency-averaged mean intensity into the core and wing contributions. Assuming a frequency-independent source function, $S_\nu = S$, and using (14.115), the core contribution is given by

$$\overline{J}_\mathrm{c} \equiv 2\int_0^{x_\mathrm{d}} J_x\phi(x)\,dx = 2S\int_0^{x_\mathrm{d}} \phi(x)\,dx \equiv SN_\mathrm{c}, \tag{14.119}$$

where N_c is called the *core normalization*. We use a *wing normalization*:

$$N_\mathrm{w} = 1 - N_\mathrm{c} = 2\int_{x_\mathrm{d}}^\infty \phi(x)\,dx. \tag{14.120}$$

Here we begin to see an intimate relation between the core-saturation method and the Osterbrock picture. Setting the parameter γ to 1, the escape probability in the Osterbrock picture is given by

$$P(\tau) = \tfrac{1}{2}N_\mathrm{w}. \tag{14.121}$$

Now, using the notion of wing normalization, we can express the net radiative rate as

$$n_jA_{ji} + n_jB_{ji}\overline{J}_{ij} - n_iB_{ij}\overline{J}_{ij} = n_jA_{ji} + (n_jB_{ji} - n_iB_{ij})[S_{ij}(1 - N_\mathrm{w}) + \overline{J}_\mathrm{w}]. \tag{14.122}$$

At any depth in the medium, divide frequency space into two parts, the *core* and the *wing*. To this end, choose a parameter $\gamma \approx 1$ such that in the core region, $\tau_\nu \geq \gamma$. Then make the approximation that

$$I_\nu = S_\nu \quad \text{for} \quad \tau_\nu \geq \gamma. \tag{14.115}$$

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For instance, for a Doppler profile we have, in analogy to equation (14.29a),

$$x_d = \sqrt{\ln(\tau/\gamma\pi^{\frac{1}{2}})}. \tag{14.117}$$

The next step is to write down the corresponding expression for the net radiative rate. First, we express the frequency-averaged mean intensity $\bar{J}$ as

$$\bar{J} = 2\int_0^\infty J_x\phi(x)\,dx = 2\int_0^{x_d} J_x\phi(x)\,dx + 2\int_{x_d}^\infty J_x\phi(x)\,dx \equiv \bar{J}_c + \bar{J}_w, \tag{14.118}$$

i.e., we split the frequency-averaged mean intensity into the core and wing contributions. Assuming a frequency-independent source function, $S_\nu = S$, and using (14.115), the core contribution is given by

$$\bar{J}_c \equiv 2\int_0^{x_d} J_x\phi(x)\,dx = 2S\int_0^{x_d}\phi(x)\,dx \equiv SN_c, \tag{14.119}$$

where N_c is called the *core normalization*. We use a *wing normalization*:

$$N_w = 1 - N_c = 2\int_{x_d}^\infty \phi(x)\,dx. \tag{14.120}$$

Here we begin to see an intimate relation between the core-saturation method and the Osterbrock picture. Setting the parameter γ to 1, the escape probability in the Osterbrock picture is given by

$$P(\tau) = \tfrac{1}{2}N_w. \tag{14.121}$$

Now, using the notion of wing normalization, we can express the net radiative rate as

$$n_jA_{ji} + n_jB_{ji}\bar{J}_{ij} - n_iB_{ij}\bar{J}_{ij} = n_jA_{ji} + (n_jB_{ji} - n_iB_{ij})[S_{ij}(1-N_w) + \bar{J}_w]. \tag{14.122}$$

So far, the expression is exact. Assuming further that the total source function is given by the line source function (i.e., we neglect the contribution from a continuum opacity as well as an overlap of other lines), the source function is given by

$$S = S_{ij} = \frac{n_jA_{ji}}{n_iB_{ij} - n_jB_{ji}}. \tag{14.123}$$

Substituting (14.123) into (14.122), we are left with

$$n_jA_{ji} + n_jB_{ji}\bar{J} - n_iB_{ij}\bar{J} = n_jA_{ji}N_w + (n_jB_{ji} - n_iB_{ij})\bar{J}_w. \tag{14.124}$$

This equation also has a profound significance in NLTE radiative transfer. The expression for the net radiative rate is very similar to the original one, the difference being that the spontaneous emission term is multiplied by the wing normalization N_w, and the net absorption term contains the wing part of the frequency-averaged mean intensity $\bar{J}_w$ instead of $\bar{J}$. Physically, this result corresponds to the following picture: a photon created in a given line has a largest probability to be created with frequencies near the line core. Such a photon travels a short distance (since the opacity it "sees" is large), until it is absorbed. When it is re-emitted, it is emitted again most likely with a frequency close to the line center. Only in the rare event when the photon has a sufficiently large frequency separation from the core can it travel a large distance in physical space or escape altogether from the medium. This was already mentioned in § 14.2 and depicted schematically in figure 14.3. In other words, roughly speaking, the core frequencies are inefficient for line transfer, and the only frequency region that is mainly responsible for a transfer of line radiation is the line wing.

Another closely related view is that (14.124) introduces the concept of *preconditioning* of the rate equations. Typically, the net rate is given by a difference of two large terms that nearly cancel. For instance, deep in the medium, most absorptions are balanced by emissions, more or less at the same spot (or, if not, very close to the original spot; see the previous section). Such a situation is very unfavorable for any iterative numerical method, since a small error in either of the two rates may lead to disastrously large errors in the current value of the *net* rate. The idea of preconditioning is to remove analytically the large contributions that balance each other and leave only the active terms.

This is exactly what was accomplished by the core-saturation method. Then, instead of having a difference of two large terms, the total emission and absorption rate, we have a difference between two "effective" transition rates. Taking again deep layers, the wing normalization is very small, $N_w \ll 1$, because most of the line profile is optically thick. The effective spontaneous emission rate, $n_jA_{ji}N_w$, which is thus much smaller than the total spontaneous rate, n_jA_{ji}, reflects the number of transitions that are *not* immediately balanced by inverse transitions. The same applies for the absorption rate: we may therefore say that the rates are *preconditioned*.

Generally, the important point is that the core components, which we know are inefficient for line transfer, were completely eliminated from the problem. But, at the same time, the form of the rate equation is unchanged. However, one should bear in mind that although elegant and intuitively clear, the preconditioning based on the core-saturation method is only *approximate*. We consider a more exact, although conceptually similar, version of the preconditioning of the rate equations below in § 14.5.

The method outlined by Rybicki [918, 919] may be used as an approximate numerical method for solving a NLTE line transfer problem. Rybicki has also suggested an iterative extension of the method that would allow one to obtain an *exact* solution [918]. Nevertheless, in view of modern, fast, and accurate numerical schemes like the ALI method, this method is not used any longer in current computational work. It is the *concept* of core saturation that has significant value for understanding the basic physics of line transfer. As was pointed out in chapter 13, the core-saturation method was one of the basic inspirations of the early versions of the ALI scheme [955]; in fact, the early approximate Λ-operators were based on the core-saturation idea.

14.4 EQUIVALENT-TWO-LEVEL-ATOM APPROACH

Classical Scheme

As we have seen in § 14.2, a coupling the kinetic equilibrium equations and the radiative transfer equation is easy to handle for a two-level atom, because in that case the kinetic equilibrium equation can easily be eliminated. One obtains a single integral equation for the source function. A straightforward application of this idea to the case of multi-level atoms consists of selecting a single transition, say, $l \leftrightarrow u$, for which the coupling of transfer and kinetic equilibrium equations is treated explicitly, while for the remaining transitions one assumes that the level populations and the radiation field are known. In other words, only two selected levels, l and u, are directly coupled to the radiation field in the transition $l \leftrightarrow u$, so the formalism will resemble that of a two-level atom. The procedure is called the *equivalent-two-level-atom* (ETLA) method.

Before more powerful and robust numerical schemes for treating a general multi-level atom problem were developed, in particular, those based on the application of the ALI scheme, the ETLA method was popular. But one can immediately see the drawback of this approach: because the information about other transitions is lagged, and in fact the other transitions are treated essentially by a Λ-iteration, the overall process may converge very slowly or not converge at all.

Nevertheless, it is still being used in several popular computer codes such as PANDORA [68–70, 1118] and ALTAIR [200]. It is also successful for line transfer in expanding atmospheres [740], as discussed in § 19.6, and some ideas based on ETLA are useful in treating lines transfer in multi-level atoms with partial frequency redistribution (see chapter 15).

The kinetic equilibrium equations for levels l and u are written as

$$n_l(R_{lu} + a_1) = n_u(R_{ul} + C_{ul}) + a_2, \tag{14.125a}$$

$$n_u(R_{ul} + a_3) = n_l(R_{lu} + C_{lu}) + a_4, \tag{14.125b}$$